STABILITY ANALYSIS OF NUCLEAR REACTORS USING LIAPUNOV'S SECOND METHOD

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THESIS

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ABSTRACT

Any change in neutronic properties in a reactor operating at steady state will result in a change in the equilibrium neutron flux and hence, the power of the reactor. A main cause for a change in neutronic properties is the high temperature attained in a reactor, which produces a feedback in the reactor operation. The response of the reactor to a particular feedback is analyzed by using Liapunov's Second Method to specify stability regimes. Both, the point-kinetics model and the distributed parameters system are analyzed.

Data from a typical thermal and fast reactors is used to specifically determine the stability domains.

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TABLE OF SYMBOLS

B^2	reactor buckling
C_f	heat capacity of the fuel = $Cp_f \cdot \rho_f$
$\mathcal{C}$	delayed-neutron precursors concentration
$\mathscr{C}$	change of delayed-neutron precursors concentration from steady state
D	neutron diffusion coefficient
g	fast non-leakage probability
k	neutron multiplication factor
ℓ	effective neutron lifetime
L	neutron diffusion length
p	resonance escape probability
P	power
$\mathscr{P}$	power removal
t	time
T	temperature
v	average neutron lifetime
ω	width of the slab reactor
W	dimensionless reactor half-width
x	spatial coordinate
y	dimensionless spatial coordinate
α	inverse thermal non-leakage probability
β	delayed neutron fraction
γ	temperature coefficient of reactivity
ε	average energy released per fission
θ	change of temperature from steady state

λ	decay constant of precursor
ν	number of fast neutrons per fission
Σ_a	macroscopic absorption cross section
Σ_f	macroscopic fission cross section
τ	dimensionless time
ϕ	neutron flux
ψ	change of neutron flux from steady state

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I. LIAPUNOV'S SECOND METHOD

A. HISTORY

In 1892 the Russian mathematician A. M. Liapunov published a long paper dealing with the problem of stability of motion (1). This paper was translated into French in 1907 and reprinted in America in 1949. Liapunov's Theory received by then only little attention due to the difficulty in understanding the advanced mathematical theorems, to the abstract way it was presented and to its lack of practical application and for a long time it was nearly forgotten. About 35 years ago, Soviet mathematicians resumed the investigation of Liapunov's Theory and its excellent application in several technical fields, mainly in control engineering, was noticed. Significant work in this area was published by Malkin (2), Letov (3), Lur'e (4) and Chetaev (5). The excellent paper by Massera (6) and the translation to English of most of the Russian works stirred up considerable curiosity in this country. Bellman (7) in 1953 published an excellent work concerning stability, but the section on Liapunov's method is difficult to comprehend. According to this author, Hahn (8), Krasovskii (9) and LaSalle and Lefschetz (10) have the best treatises on the subject, the consolidation of the concepts of Liapunov's Theory and the clear presentation of it, make these three books the main references for workers in the field.

Recently, Zubov (11) found the best extension of the Liapunov's Theory, to include the analysis of partial differential equations, which is by now the main topic for research. Yet, a good amount of work remains to be done. All the previous works, including Liapunov's original theory, were devoted to the analysis of stability of ordinary differential equations.

Many authors have applied Zubov's extension to works concerning vibrations, reactor physics, hydrodynamics, magnetohydrodynamics and control processes. The list of references will provide the interested reader with the main works in this field.

B. THEORY

The name "second method" (or "direct method") is of historical origin. Liapunov's Theory is divided in two categories. The "first method" which comprises all procedures in which the explicit form of the solutions is used, specially when represented by infinite series and the investigation of stability "in the small" (local stability) studies the singular points of a nonlinear differential equation by using the appropriate linearized version of the differential equation near the singular point. The "second method" attempts to make stability statements by using (in addition to the differential equations) suitable functions, called Liapunov functions, which are defined in the motion space. This method which deserves special study due to

its inherently advantages over most of the conventional methods, investigates the stability "in the large" (global stability).

Liapunov's Second Method is in essence a more general expression of the Hamiltonian (total energy), and it is based in the statement that a physical system loses potential energy in a neighborhood of a point of stable equilibrium. It is said that it is more general than the total energy method, because unlike the energy of a system, the Liapunov function, denoted V, is not unique. This is the main reason why the "second method" is a tool in the analysis of stability of dynamical systems, which must be used with considerable skill. One of the main features of this method is its appeal to geometric intuition; $V(\bar{x},t)$ can be seen as a measure of the "distance" of the state $(\bar{x},t)$ from the origin, in the state space and the variation of this "distance" (norm of the differential equation) as t varies will provide definite bounds of stability regions for the prescribed system under consideration.

Normally stability with respect to one norm does not imply stability with respect to another norm. This difficulty does not appear in finite-dimensional systems since all norms defined on a finite-dimensional vector space are equivalent.

1. Fundamental Concepts and Definitions

Stability is a property of certain systems of differential equations and is basically concerned with the

question of whether or not the dynamic system will return to a particular state after it has been disturbed in some way.

Differential equations in their most general way can be expressed as

$$\frac{\partial U(\bar{x},t)}{\partial t} = F[U(\bar{x},t),t]$$

where $U(\bar{x},t)$ is a multidimensional function of space and time. It may well happen that F depends upon $\bar{U}(t)$ alone and not upon time explicitly. Then the previous equation assumes the simpler form

$$\dot{U}(t) = F[U(t)]$$

A system of this nature is known as autonomous. Since it is not intended in this work to expose the reader with the mathematical aspect of Liapunov's Theory, in general, the presentation of the theory will be made without proofs. Let us denote $\Omega(R)$ the spherical region $||x|| < R$ and by $\Lambda(R)$ the sphere boundary $||x|| = R$. The matter of concern is the stability of the origin. Initiating the motion at a point x_o, the origin is said to be:

(a) Stable whenever the path remains in the spherical region $\Omega(R)$; that is,the path never reaches the boundary sphere $\Lambda(R)$, Figure 1.

(b) Asymptotically stable whenever it is stable and in addition the path tends to the origin as time increases indefinitely.

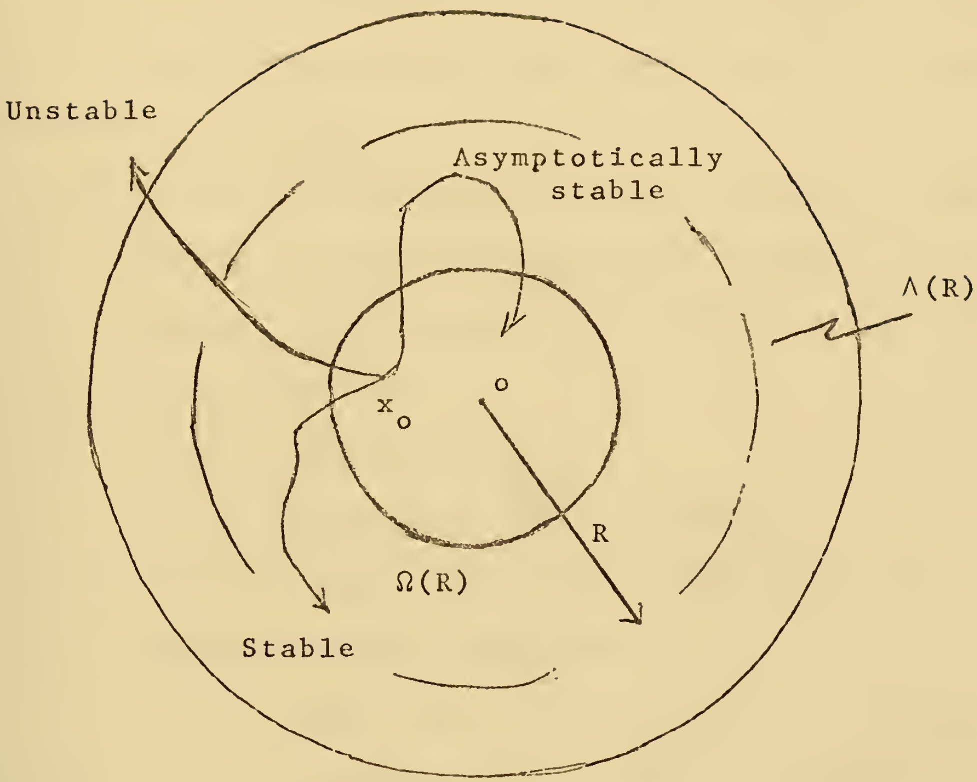

Figure 1. Domains of stability.

(c) Unstable whenever the path reaches the boundary sphere $\Lambda(R)$.

For autonomous systems:

A Liapunov function, $V(U)$, is defined as a positive definite scalar function with the following properties:

(a) $V(U)$ is continuous with its first partial derivatives in a certain open region about the origin

(b) $V(o) = 0$ only

(c) Outside the origin and always inside the open region, V is non-negative and vanishes only at the origin. The origin is an isolated minimum of V; and in addition

$\dot{V} \leq 0$ in the open region

If $\dot{V} = 0$, the stability is neutral

If $\dot{V} < 0$, the stability is asymptotic.

For nonautonomous systems:

Let us define $V_1(U)$ as previously. A Liapunov function, $V(U,t)$ is defined as a positive definite scalar function with the following properties:

(a) $V(U,t)$ is defined in the open region for all $t \geq 0$.

(b) $V(o,t) = 0$ for $t \geq 0$.

(c) $V_1(U) \leq V(U,t)$ for all x in the open region and all $t \geq 0$; and in addition

$\dot{V} \leq 0$ in the open region

where

$$\boxed{\dot{V}(U,t) = \frac{\partial V}{\partial t} + U \cdot \nabla V}$$

Summarizing these conditions, the three main steps to test if a scalar function is a Liapunov function are:

1. $V > 0$
2. $V(o) = 0$
3. $\dot{V} \leq 0$

2. Methods to Construct a Liapunov Function

Several authors have attempted to present guidelines for generating Liapunov Functions. Unfortunately, most of these methods are extremely restrictive, limitations being imposed primarily by the number of nonlinear terms and the order of the system. Ingwerson (12) presents a Table to generate Liapunov Functions for linear autonomous differential equations up to the fourth order. Barbashin (13) and Lur'e and Rozenvasser (14) tried to establish some rules for construction of Liapunov Functions. Schultz (15) has one of the best methods available now; it is called the Variable Gradient Method, being the least restrictive for the case of autonomous systems. Its only restriction is that all nonlinearities must be single-valued.

In general, literature late in 1961 and 1962 became saturated with methods for generating Liapunov Functions, most of them with the restrictions previously mentioned. Furthermore, such methods, when applicable, were directed to autonomous systems only.

Although development of Liapunov stability theory and applications to ordinary differential equations has progressed rapidly, its application to partial differential equations has remained limited. The importance of partial differential equations in the fields of reactor physics, hydrodynamics, control processes, etc. has motivated investigations of possible ways to extend Liapunov stability theory to partial differential equations.

In this study Functional Analysis plays an important role, because the stability of the equilibrium solution is defined in terms of the norm induced by the inner product of the Hilbert space on which the solutions of the system are defined. Thus, with proper choice of inner product or norm, the square of the norm becomes the Liapunov functional which establishes asymptotic stability. In fact what is being done is the construction of a scalar function of the distance between the solution and the equilibrium point of interest. If this distance function, evaluated along the solution paths, obeys certain inequalities, then statements concerning the stability of the equilibrium point can be made. Movchan (16) defines stability in terms of two metrics, rather than one, to be more restrictive on the initial states. The choice of the initial state space and the metric is crucial in the formulation of stability problems in the framework of Liapunov's Second Method. For cases in which only the behavior of some of the state variables is of importance or some function of the state variables is of interest to the analyst, then, for these cases, it is meaningful to define Liapunov stability with respect to two metrics. For instance, one may be only interested in the behavior of the maximum deflections in an elastic system, regardless of the velocity and acceleration involved into attained it.

According to Kastenberg & Ziskind(17) extreme care has to be taken when interpreting the obtained results,

because severe peaking in some of the state variables of the system can cause the L_2 norm of the system trajectory to move out of the domain. Wang (18) and Parks (19) have devoted special attention to the study of panel flutter which represents a linear distributed parameter system. It is concluded that a good comprehension of Hilbert and Banach spaces will provide the analyst with an excellent tool for generating Liapunov functionals.

3. Comparison with Other Methods

The advantages of Liapunov's Second Method over the conventional methods used for stability analysis can be summarized as follows:

(a) It employs the system equations without resorting to approximations. Normally a distributed system is approximated by a lumped parameter model having finite or infinite number of degrees of freedom. This method is not satisfactory because quite often it exhibits characteristics which do not agree with the physical nature of the problem. Liapunov's method is in general a more reliable method.

(b) There is no theoretical limit on the number of nonlinearities or on the order of the differential equation to which Liapunov's Second Method can be applied.

(c) The laborious work implied in finding the system solution is avoided. Normally the system equations are integrated numerically for some given perturbations in the initial conditions. This method is time consuming and

sometimes does not give an accurate presentation of the system behavior.

(d) The relationship between the system stability and the system parameters is directly extracted from the analysis of Liapunov's method conditions for stability.

(e) There are no mathematical restrictions with respect to uniqueness.

The method presents also some deficiencies; among them the main ones are:

(a) The construction of the required Liapunov Function for the system under consideration is without doubt the most difficult part of the task due to the fact that there is no guideline to construct it and success depends mainly upon the ingenuity and experience of the analyst.

(b) The interpretation of the obtained results has to be done carefully.

(c) Integral inequalities play an important role in deriving conditions for stability. As a consequence, the regions of stability may be somewhat looser than those obtained by other methods.

In conclusion, in spite of the difficulties cited above, the advantages of the Liapunov's Second Method makes it an excellent method for studying stability.

C. APPLICATION TO NUCLEAR REACTOR CONTROL

Liapunov's Second Method has been applied with success by Hsu (20) who considers the linear and non-linear cases,

analyzing them by means of the Spectral Analysis and Liapunov's Method, and comparing the results obtained by each method. Hsu shows that the Spectral Analysis provides a little more information than Liapunov's Method, but this latter eliminates the calculation of the eigenvalues of the system, and for the nonlinear system analysis bounds of stability are presented.

Kastenberg (21) presents a clear comparison of Liapunov's Second Method with the Semigroup Method, and the Comparison Function and Maximum Principle Techniques. In the Liapunov and Semigroup methods, one must obtain an a priori bound on the system nonlinearity and then proceed. When employing the maximum principle or the comparison function technique, an a priori bound on the system nonlinearity is not always required. In contrast, one must give a bound on the initial condition. For cases which are stable in a global sense, the results of the various methods coincide.

The application of Liapunov's Second Method to nuclear reactor control seems to be excellent, mainly for the spatially-dependent reactor system, due to the fact that the method allows the inclusion of any number of nonlinearities. This will permit the study of spatial effects of temperature, the main feedback effect in a reactor, control rod motion and several other processes within a reactor. Also, the inclusion in the analysis of the different reactivity coefficients, as doppler resonance broadening and structural expansion is allowed.

As pointed before, the interpretation of the results must be done very carefully due to the fact that a Liapunov Function is not unique.

II. MATHEMATICAL MODEL OF THE REACTOR SYSTEM

Both Thermal and Fast reactors are described essentially by the same basic dynamic principles, regardless of material and geometry considerations.

For a given reactor size, the reactor reactivity depends on the neutron cross-sections and on the relative amounts and densities of different materials. All of these being affected by the temperature, the reactivity will be strongly coupled to the power of the reactor and the reactor governing equations become nonlinear.

The Thermal Reactor stability is analyzed by means of the lumped-parameter (point-kinetics) model, in which the partial differential equations are reduced to ordinary differential equations via spatial discretization. This is justified only when infinitesimal small perturbations are introduced, as in the case when k is very near to unity (very small departures from "critical"). Thermal reactors are generally smaller in size than fast-breeder reactors. The Fast-Breeder Reactor stability is analyzed using the governing partial differential equations without resorting to any type of approximations. Due to possible stronger space effects in fast reactors than in thermal reactors, the space and time of the state variables of the system must be maintained during the analysis. The reactivity insertion in the system, Figure 2, can be either positive or negative. Both cases will be considered in this work.

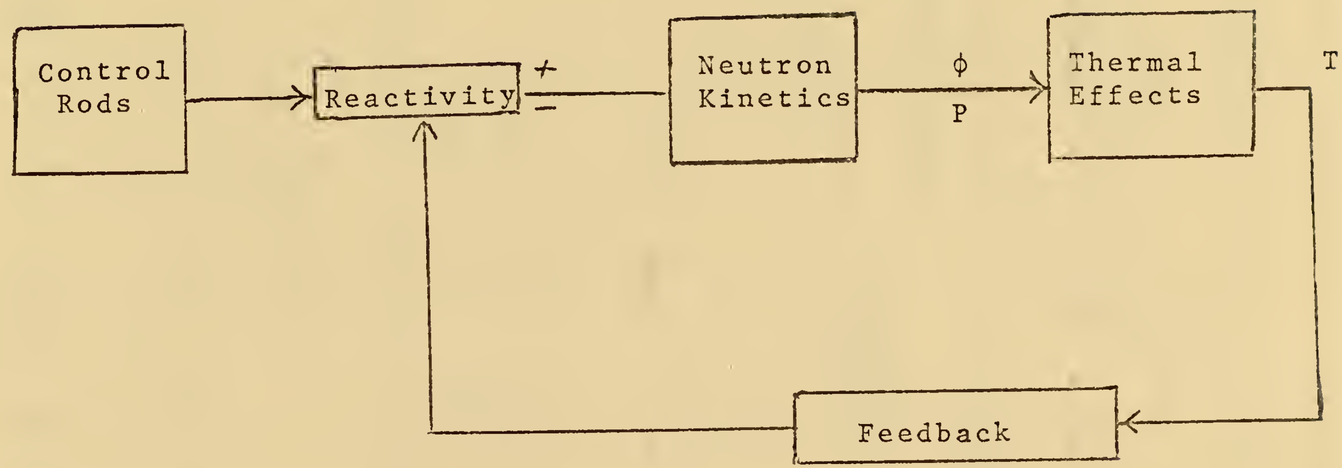

Figure 2. Feedback effects diagram.

A. POINT-KINETICS MODEL

1. Diffusion Equation

According to Meghreblian and Holmes (22), the neutron diffusion equation is written in the following way:

$$D\,\nabla^2\phi(\bar{r},t) - \Sigma_a\phi(\bar{r},t) = \frac{1}{v}\frac{\partial\phi(\bar{r},t)}{\partial t} -$$
$$- (1-\beta)\nu\,\Sigma_f\, pg\,\phi(\bar{r},t) - pg\sum_i \lambda_i\, C_i(\bar{r},t) \qquad (1)$$

Expressing the equation for a slab reactor and using only one-delayed neutron group:

$$D\,\frac{\partial^2\phi(x,t)}{\partial x^2} - \Sigma_a\phi(x,t) = \frac{1}{v}\frac{\partial\phi(x,t)}{\partial t} -$$
$$- (1-\beta)\nu\,\Sigma_f\, pg\,\phi(x,t) - pg\,\lambda\, C(x,t) \qquad (2)$$

It is looked for a solution which is separable in time and space:

$$\phi(x,t) = \phi(t)\,F(x) \qquad (a) \qquad (3)$$
$$C(x,t) = C(t)\,F(x) \qquad (b)$$

Applying relations (3) to Equation (2) yields

$$\frac{D}{F(x)}\,\frac{\partial^2 F(x)}{\partial x^2} - \Sigma_a = \frac{1}{\phi(t)}\left[\frac{1}{v}\,\dot{\phi}(t) - \right.$$
$$\left. - (1-\beta)\nu\,\Sigma_f\, pg\,\phi(t) - pg\,\lambda\, C(t)\right] \qquad (4)$$

Thus

$$\frac{1}{F(x)}\,\frac{\partial^2 F(x)}{\partial x^2} = \frac{1}{D}\left\{\Sigma_a + \frac{1}{\phi(t)}\left[\frac{1}{v}\,\dot{\phi}(t) - \right.\right.$$
$$\left.\left. - (1-\beta)\nu\,\Sigma_f\, pg\,\phi(t) - pg\,\lambda\, C(t)\right]\right\} = -B^2 \qquad (5)$$

From Equation (5) it is obtained that:

$$\frac{\partial^2 F(x)}{\partial x^2} + B^2 F(x) = 0 \qquad \text{(a)}$$

$$(6)$$

$$(DB^2+\Sigma_a)\phi(t) = -\frac{1}{v}\dot{\phi}(t)+(1-\beta)\nu\Sigma_f pg\phi(t)+pg\ \lambda\ C(t) \qquad \text{(b)}$$

Using the following relationships:

$$L^2 = \frac{D}{\Sigma_a} \qquad k = \frac{\nu\ \Sigma_f\ pg}{\Sigma_a(1+L^2B^2)} \qquad \ell = \frac{1}{v(\Sigma_a+DB^2)}$$

and $k = \Delta k + 1$, Equation (6b) becomes

$$\ell\dot{\phi}(t) = [(1-\beta)\Delta k-\beta]\phi(t) + \frac{pg}{\Sigma_a+DB^2}\ \lambda\ C(t) \qquad (7)$$

Let P(t) be the time behavior of the reactor power generation

$$P(t) = \varepsilon\ \Sigma_f\ \phi(t) \qquad (8)$$

then $\dot{P}(t) = \varepsilon\ \Sigma_f\ \dot{\phi}(t)$, and the equation describing the reactor power is

$$\ell\dot{P}(t) = [(1-\beta)\Delta k-\beta]P(t) + \frac{\varepsilon\ \Sigma_f\ pg}{\Sigma_a+DB^2}\ \lambda\ C(t) \qquad (9)$$

It will be seen in the next paragraph that for certain kinetics problems a simplification can be obtained by assuming the infinite delay time in delayed neutron emission:

$$\lambda C(o) = \frac{\nu\beta}{\varepsilon}\ P(o)$$

This expression is used as a first estimate for the precursor concentration.

Then

$$\boxed{\ell \dot{P}(t) = [(1-\beta)\Delta k-\beta]\ P(t) + [1+\Delta k]\ \beta\ P(o)} \qquad (10)$$

2. <u>Concentration of Precursors Equation</u>

$$\frac{\partial}{\partial t} C_i(\bar{r},t) = -\lambda_i\ C_i(\bar{r},t) + \nu\beta_i\ \Sigma_f\ \phi(\bar{r},t) \qquad (11)$$

For a slab reactor and using one-delayed neutron group, Equation (11) can be expressed as

$$\frac{\partial}{\partial t} C(x,t) = -\lambda\ C(x,t) + \nu\ \beta\ \Sigma_f\ \phi\ (x,t) \qquad (12)$$

Using Equations (3) to separate variables, Equation (12) becomes

$$\dot{C}(t) = -\lambda\ C(t) + \nu\ \beta\ \Sigma_f\ \phi(t) \qquad (13)$$

Then, using Equation (8) yields

$$\boxed{\dot{C}(t) = -\lambda\ C(t) + \frac{\nu}{\varepsilon}\ \beta\ P(t)} \qquad (14)$$

From Equation (14), the steady state concentration of the precursors can be obtained as:

$$\lambda C(o) = \frac{\nu\beta}{\varepsilon}\ P(o) \qquad (15)$$

This result was previously used.

3. Energy Equation

Expressing $H(\bar{r},t)$ as the energy content per unit volume, then the time rate of change of H must be given by the net energy gain per unit time and volume from the fission reactions and the reactor cooling:

$$H(\bar{r},t) = \rho C_p [T(\bar{r},t) - T_o(\bar{r})] = \rho C_p \; \theta(\bar{r},t) \tag{16}$$

and

$$\frac{\partial}{\partial t} H(\bar{r},t) = \varepsilon \; \Sigma_f \; \phi(\bar{r},t) - \mathcal{P}(\bar{r},t) \tag{17}$$

For a slab reactor, the energy balance becomes

$$\rho C_p \frac{\partial}{\partial t} \theta(x,t) = \varepsilon \; \Sigma_f \; \theta(x,t) - \mathcal{P}(x,t) \tag{18}$$

Let

$$\theta(x,t) = \theta(t) \; F(x) \qquad \text{(a)} \qquad (19)$$

$$\phi(x,t) = \phi(t) \; F(x) \qquad \text{(b)}$$

$$\mathcal{P}(x,t) = \mathcal{P}(t) \; F(x) \qquad \text{(c)}$$

Then

$$\rho C_p \; \dot{\theta}(t) = \varepsilon \; \Sigma_f \; \phi(t) - \quad (t) \tag{20}$$

Let $C_f = \rho C_p$ and using Equation (8), Equation (20) becomes

$$\boxed{C \; \dot{\theta}(t) = P(t) - \mathcal{P}(t)} \tag{21}$$

Three cases are normally encountered for the power removal:

(a) Adiabatic, $\mathcal{P}(t) = 0$

(b) Constant power removal, $\mathcal{P}(t) = A$

(c) Newton's law of cooling, $\mathcal{P}(t) = h\theta(t)$

4. <u>Feedback Models</u>

a. Temperature Feedback Model #1

The multiplication factor is temperature and time dependent

$$k = k(T) = k[T(t)] \tag{22}$$

Expanding it in powers of $[T(t) - T_o]$, yields

$$k(T) = \sum_{n=0}^{\infty} \frac{dk^n(T)}{dt} [T(t) - T_o]^n$$

thus

$$k(T) = k_o \; [1 + \sum_{n=0}^{\infty} An \; (T - T_o)]^n \tag{23}$$

which becomes, upon dropping all terms beyond $n = 1$,

$$k = k_o[1 + \gamma(T-T_o)] = (1 + \Delta k_o)(1 + \gamma\theta) \tag{24}$$

or

$$\boxed{\Delta k(t) = \Delta k_o + \gamma\theta(t)} \tag{25}$$

where Δk_o is the change in k applied to the reactor at $t = 0$ and γ represents the temperature coefficient of reactivity, which could be either (+) or (-).

b. Feedback Model #2

One of the major considerations to be made when accounting for the temperature dependence of the multiplication is that described by the nuclear Doppler effect.

Thompson and Beckerley (<u>23</u>) expresses that dependence as

$$\frac{dk}{dT} = \gamma_D(\frac{300}{T})^n \tag{26}$$

where T is in °K.

From (26), Δk can be expressed explicitly as

$$\boxed{\Delta k = \Delta k_o + \frac{\gamma_D}{1-n}[T(\frac{300}{T})^n - T_o(\frac{300}{T_o})^n]} \qquad (27)$$

being n = 1 a special case, for which a logarithmic expression is found.

Typical values of n are 1/2, 1 and 3/2.

B. DISTRIBUTED-PARAMETER REACTOR SYSTEM

1. Diffusion Equation

The diffusion equation is used in this case without resorting to any approximation, then

$$D\nabla^2\phi(\bar{r},t) - \Sigma_a\phi(\bar{r},t) = \frac{1}{v}\frac{\partial\phi}{\partial t}(\bar{r},t) -$$

$$-(1-\beta)\nu\Sigma_f pg \quad \phi(\bar{r},t) - pg\sum_i \lambda_i \; C_i(\bar{r},t) \qquad (28)$$

For a slab reactor and using one-group delayed neutron, the diffusion equation takes the following form:

$$D\frac{\partial^2\phi}{\partial x^2}(x,t) - \Sigma_a \; \phi(x,t) = \frac{1}{v} \; \frac{\partial\phi}{\partial t}(x,t) -$$

$$-(1-\beta)\nu \; \Sigma_f \; pg \; \phi(x,t) - pg \; \lambda \; C(x,t) \qquad (29)$$

The cross sections should be written in a proper way as time-dependent but these variations are assumed negligible in comparison to the rapid transients in ϕ, C and T. The equation for a non-trivial, steady state solution $\phi_o(x)$, $C_o(x)$ is:

$$D\frac{\partial^2\phi_o(x)}{\partial x^2} - \Sigma_a\phi_o(x) = -(1-\beta)\nu\Sigma_f \; pg \; \phi_o(x) - pg \; \lambda \; C_o(x) \qquad (30)$$

Now subtracting Equation (30) from Equation (29) yields

$$D\frac{\partial^2\psi}{\partial x^2}(x,t) - \Sigma_a\ \psi(x,t) = \frac{1}{v}\ \frac{\partial\psi}{\partial t}(x,t) -$$

$$-(1-\beta)\nu\ \Sigma_f\ pg\ \psi(x,t) - pg\ \lambda\ \mathcal{C}\ (x,t) \qquad (31)$$

where

$$\psi(x,t) = \phi(x,t) - \phi_o(x) \qquad (a) \qquad (32)$$

$$\mathcal{C}\ (x,t) = C(x,t) - C_o(x) \qquad (b)$$

This change of variables moves the equilibrium state of the system from $(\phi_o,\ C_o)$ to $(0,\ 0)$, Figure 3.

Define

$$k = \frac{\nu\ \Sigma_f\ pg}{\Sigma_a\ \alpha_1} \qquad (33)$$

and using

$$\alpha = 1 + L^2\ B^2 \qquad (34)$$

the diffusion equation becomes

$$D\frac{\partial^2\psi(x,t)}{\partial x^2} - \Sigma_a\ \psi(x,t) + (1-\beta)k\ \Sigma_a\ \alpha\ \psi(x,t) +$$

$$+\ pg\ \lambda\ \mathcal{C}\ (x,t) = \frac{1}{v}\ \frac{\partial\psi(x,t)}{\partial t} \qquad (35)$$

Dimensionless space and time expressions are introduced at this point:

$$y = \frac{x}{L} \qquad (x >> L) \qquad (a) \qquad (36)$$

$$\tau = \nu\Sigma_a t \qquad (b)$$

Then Equation (35) becomes

$$\boxed{\frac{\partial^2\psi(y,\tau)}{\partial y^2} + [(1-\beta)k\alpha-1]\psi(y,\tau) + \frac{k\alpha}{\nu\Sigma_f}\ \lambda\ \mathcal{C}\ (y,\tau) = \frac{\partial\psi(y,\tau)}{\partial\tau}} \qquad (37)$$

The boundary conditions are: (a) $\psi(W,\tau) = 0$ (38)

(b) $\psi(-W,\tau) = 0$

where $W = \dfrac{x_o}{2L}$ expresses the dimensionless half-width of the

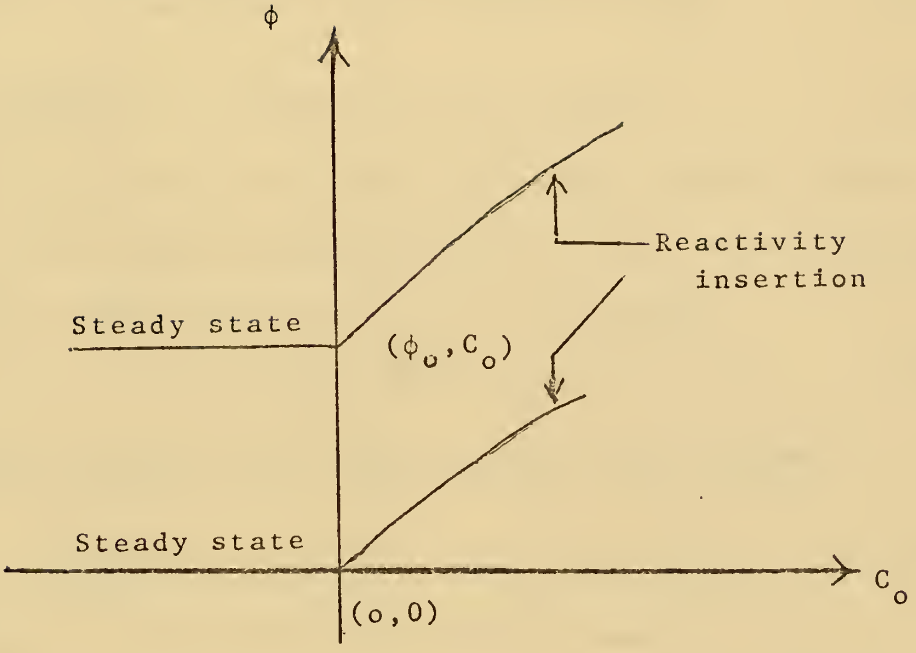

Figure 3. Translation of equilibrium states.

slab reactor including the extrapolation distance. The initial conditions are: (a) $\psi(y,o) = 0$ (39)

(b) $\mathcal{C}(y,o) = 0$

2. Concentration of precursors equation

$$\frac{\partial C_i(\bar{r},t)}{\partial t} = -\lambda_i C_i(\bar{r},t) + \nu\beta_i \Sigma_f \phi(\bar{r},t) \quad (40)$$

For a slab reactor and using one-group delayed neutron, Equation (40) is written in the following way:

$$\frac{\partial C(x,t)}{\partial t} = -\lambda C(x,t) + \nu\beta\, \Sigma_f\, \phi(x,t) \quad (41)$$

The same assumption used in the diffusion equation, with respect to the cross sections is used here. A non trivial, steady state solution $\phi_o(x)$, $C_o(x)$ is assumed:

$$0 = -\lambda C_o(x) + \nu\beta\, \Sigma_f\, \phi_o(x) \quad (42)$$

Subtracting Equation (42) from Equation (41) yields

$$\frac{\partial \mathcal{C}(x,t)}{\partial t} = -\lambda \mathcal{C}(x,t) + \nu\beta\, \Sigma_f\, \psi(x,t) \quad (43)$$

Using Equations (36), Equation (43) becomes

$$\nu\Sigma_a \frac{\partial \mathcal{C}(y,\tau)}{\partial \tau} = -\lambda \mathcal{C}(y,\tau) + \nu\beta\, \Sigma_f\, \psi(y,\tau) \quad (44)$$

The initial condition for the equation is

$$\mathcal{C}(y,o) = 0 \quad (45)$$

Introducing the multiplication factor in Equation (44) yields

$$\boxed{\nu\Sigma_a \frac{\partial \mathcal{C}(y,\tau)}{\partial \tau} = -\lambda \mathcal{C}(y,\tau) + \frac{\beta\alpha\Sigma_a k}{pg} \psi(y,\tau)} \quad (46)$$

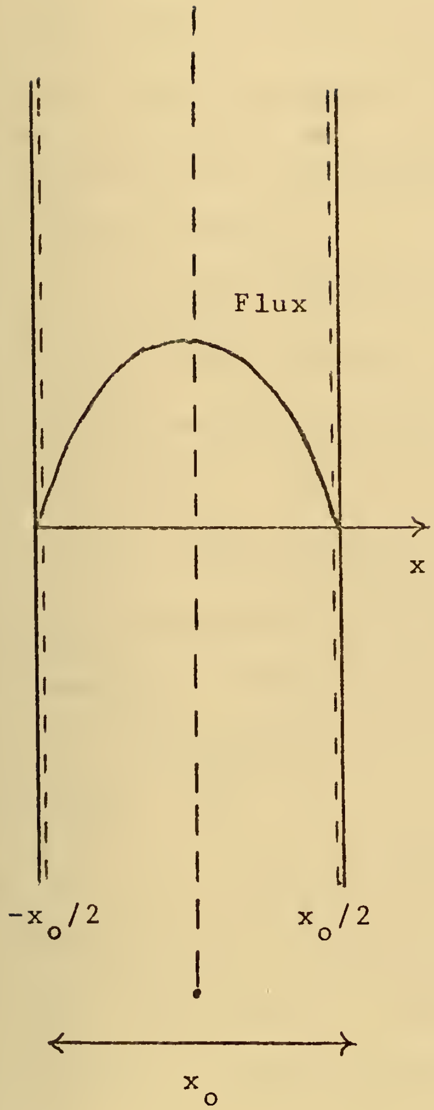

Figure 4. Dimensional slab reactor.

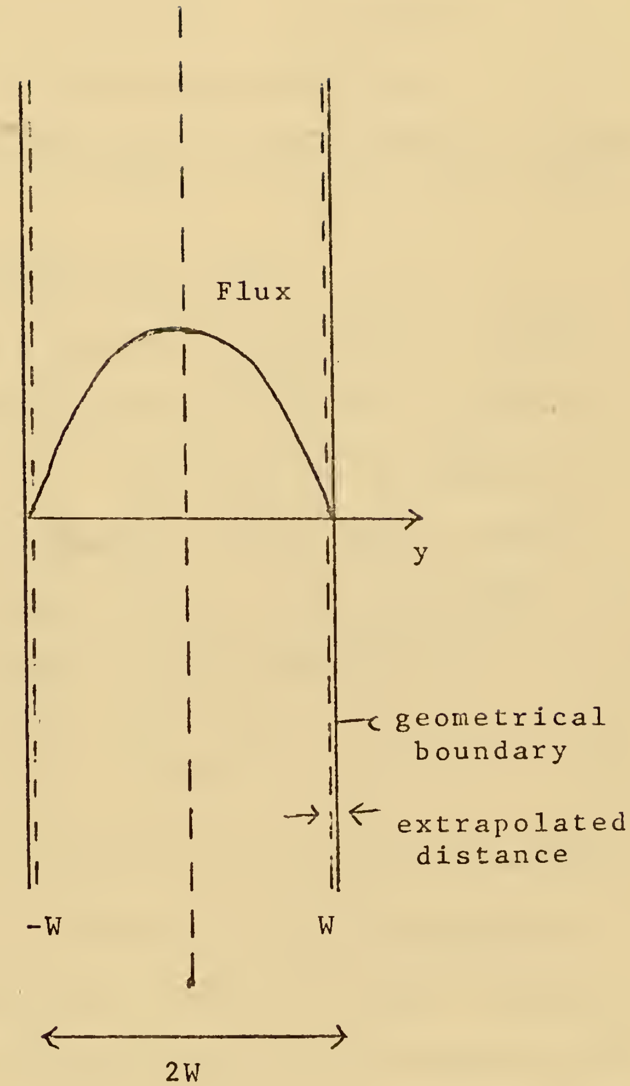

Figure 5. Dimensionless slab reactor.

3. <u>Energy Equation</u>

The energy balance condition is normally expressed as:

$$\text{Energy produced} - \text{Energy removed} = \begin{array}{c}\text{Energy stored in}\\ \text{the system}\end{array}$$

The energy produced is due to fission: $\varepsilon \, \Sigma_f \, \phi(\bar{r},t)$

The energy removed is due to the coolant: $\mathcal{P}(\bar{r},t)$

Three cases are normally encountered for the energy removed from a system:

(a) Adiabatic case, $\mathcal{P}(\bar{r},t) = 0$

(b) Constant power removal, for which the steady state requirements are normally used, $\mathcal{P}(\bar{r},o) = \varepsilon\Sigma_f\phi(\bar{r},o)$

or $\mathcal{P}(\bar{r}) = \varepsilon\Sigma_f\phi_o(\bar{r})$

(c) Newton's law of cooling, $\mathcal{P}(\bar{r},t) = h\,\theta(\bar{r},t)$

It is noticed that these cases are for stationary-fuel reactors. The energy equation can be written as:

$$\rho_f C_{p_f} \frac{\partial T_f(\bar{r},t)}{\partial t} = \varepsilon \, \Sigma_f \, \phi(\bar{r},t) - \mathcal{P}(\bar{r},t) \tag{47}$$

Let $C_f = \rho_f C_{p_f}$ be the heat capacity of the reactor system expressed as energy per unit volume per unit temperature. The subscripts f and C stand for fuel and coolant respectively. Using the same assumption as used previously with respect to the cross section, and expressing Equation (47) for a slab reactor, yields

$$\boxed{C_f \frac{\partial T_f(x,t)}{\partial t} = \varepsilon \, \Sigma_f \, \phi(x,t) - \mathcal{P}(x,t)} \tag{48}$$

a. Adiabatic Case

$$C_f \frac{\partial T_f(x,t)}{\partial t} = \varepsilon \Sigma_f \phi(x,t) \tag{49}$$

This case will not be considered in this work because it represents accident conditions.

b. Constant Power Removal

$$C_f \frac{\partial T_f(x,t)}{\partial t} = \varepsilon \Sigma_f \phi(x,t) - \mathcal{P}_o \tag{50}$$

Assuming a steady-state, nontrivial solution $\phi_o(x)$ it becomes

$$0 = \varepsilon \Sigma_f \phi(x,t) - \mathcal{P}_o \tag{51}$$

Subtracting Equation (51) from Equation (50) yields

$$C_f \frac{\partial \theta(x,t)}{\partial t} = \varepsilon \Sigma_f \psi(x,t) \tag{52}$$

where $$\theta(x,t) = T_f(x,t) - T_{fo}(x) \tag{53}$$

Introducing Equations (36), Equation (52) becomes

$$C_f v \Sigma_a \frac{\partial \theta}{\partial \tau}(y,\tau) = \varepsilon \Sigma_f \psi(y,\tau) \tag{54}$$

with initial condition:

$$\theta(y,o) = 0 \tag{55}$$

c. Newton's Law of Cooling

$$C_f \frac{\partial T_f(x,t)}{\partial t} = \varepsilon \Sigma_f \phi(x,t) - h(T_f - T_c) \tag{56}$$

Assuming a steady-state, nontrivial solution, $\phi_o(x)$ and $T_{fo}(x)$, Equation (56) becomes

$$\boxed{C_f \frac{\partial \theta(x,t)}{\partial t} = \varepsilon \Sigma_f \psi(x,t) - h\theta(x,t)} \tag{57}$$

Note that T_c and h remain constants. This is explained by means of the one effective coolant temperature assumption. Introducing the dimensionless expressions for space and time, Equation (62) becomes:

$$\boxed{C_f \; v \; \Sigma_a \frac{\partial\theta(y,\tau)}{\partial\tau} = \varepsilon \; \Sigma_f \; \psi(y,\tau) - h\theta(y,\tau)} \tag{58}$$

with initial condition: $\theta(y,o) = 0$ (59)

4. <u>Feedback Models</u>

a. Temperature Feedback Model #1

Through the temperature, the multiplication factor is now dependent on both space and time.

$$k = k(T) = k[T(x,t)] \tag{60}$$

Expanding it in powers of $[T(x,t) - T_o(x)]$ yields:

$$k(T) = \sum_{n=0}^{\infty} \frac{\partial^n k(T)}{\partial T^n} [T(x,t) - T_o(x)]^n$$

Thus

$$k(T) = k_o[1 + \sum_{n=1}^{\infty} A_n(T - T_o)^n] \tag{61}$$

which upon dropping all terms beyond n=1, reduces to:

$$k = k_o(1 + \gamma(T - T_o)] = k_o(1 + \gamma\theta) \tag{62}$$

where $\gamma = \partial k/\partial T = A_1$

and $\theta(x,t) = T(x,t) - T_o(x)$

Then, Equation (62) becomes

$$\boxed{k = 1 + \Delta k_o + \gamma\theta} \tag{63}$$

having neglected the term $\gamma \Delta k_o \theta$. The multiplication factor can now be written as:

$$k[\theta(y,\tau)] = 1 + \Delta k_o \pm \gamma\theta(y,\tau) \tag{64}$$

γ is the temperature coefficient of reactivity, which could be either (+) or (-).

b. Feedback Model #2

The temperature dependence of the multiplication factor can be expressed in a general form, adding the contributions due to the Doppler effect, due to structural expansions and due to other effects, as:

$$\frac{dk}{dT} = \gamma_D \left(\frac{a_1}{T}\right)^n + \gamma_E \left(\frac{T}{a_2}\right)^m + \gamma \tag{65}$$

where D and E stands for doppler and expansion respectively. k can be expressed explicitly, using the initial condition, $k = k_o$ at $T = T_o$, thus

$$\boxed{\begin{aligned} k = 1+\Delta k_o + \frac{\gamma_D}{(1-n)} \left[T\left(\frac{a_1}{T}\right)^n - T_o\left(\frac{a_1}{T_o}\right)^n\right] + \\ + \frac{\gamma_E}{1+m} \left[T\left(\frac{T}{a_2}\right)^m - T_o\left(\frac{T_o}{a_2}\right)^m\right] + \gamma_o(T - T_o) \end{aligned}} \tag{66}$$

when n = 1, the third term on the right hand side becomes:

$$\gamma_D \, a_1 \cdot \ln \frac{T}{T_o} \tag{66a}$$

It is noted that $a_1 = a_2 = 300°K$ and T is usually expressed in °K.

III. STABILITY ANALYSIS

A. POINT-KINETICS MODEL

The reactor under consideration for this analysis has the following characteristics:

(a) Thermal reactor.

(b) Homogeneous, bare, slab reactor.

(c) One-group delayed neutron.

(d) Constant power removal.

(e) Stationary - full reactor.

and normal operating conditions (no accidents) are assumed. The governing equations for this system are:

$$\ell \dot{P}(t) = [(1-\beta)\Delta k - \beta]\, P(t) + (1+\Delta k)\beta\, P(o) \qquad (10)$$

$$C\dot{\theta}(t) = P(t) - \mathcal{P}(t) \qquad (21)$$

using the following feedback models:

$$\text{(a)} \quad \Delta k = \Delta k_o + \gamma\theta \qquad (25)$$

$$\text{(b)} \quad \Delta k = \Delta k_o + \frac{\gamma_D}{1-n} \left[T\left(\frac{300}{T}\right)^n - T_o\left(\frac{300}{T_o}\right)^n\right] \qquad (27)$$

where Δk_o is a positive step insertion of reactivity by external means, such as control rod motion. The initial conditions for the system are:

(a) $P(o) = \mathcal{P}(o)$

(b) $\dot{\theta}(o) = 0$

(c) $\theta(o) = 0,\ T(o) = T_o$

For the constant power removal, $\mathcal{P}(t)$ becomes $P(o)$ and Equation (21) can be written as

$$c\dot{\theta}(t) = P(t) - P(o) \qquad (21a)$$

Assuming P(o) = 1 and introducing a dimensionless power, Equations (10) and (21a) can be expressed as

$$\ell\dot{P}(t) = [(1-\beta)\Delta k - \beta]P(t) + (1+\Delta k)\beta \qquad (10a)$$

$$c\dot{\theta}(t) = P(t) - 1 \qquad (21b)$$

1. <u>No Delayed Neutrons, No Reactivity Input</u>

The point kinetics model when $\beta = 0$ and $\Delta k_o = 0$ is reduced to

$$\ell\dot{P} = \gamma\theta P \qquad (67)$$

$$c\dot{\theta} = P - 1 \qquad (68)$$

After some algebraic manipulations, Equation (67) can be expressed as

$$\frac{d}{dt}\ \mathrm{In}\ P = \frac{\gamma\theta}{\ell} \qquad (69)$$

"Cross multiplying" Equations (68) and (69) yields

$$\dot{P} - \frac{d}{dt}\ \mathrm{In}\ P - \frac{\gamma_c}{\ell}\ \theta\dot{\theta} = 0 \qquad (70)$$

which can be written as

$$\frac{d}{dt}(P - \mathrm{In}\ P - \frac{1}{2}\,\frac{\gamma_c}{\ell}\ \theta^2) = 0 \qquad (71)$$

The expression inside the parentheses can be called V, then

$$V = P - \mathrm{InP} - \frac{1}{2}\ \frac{\gamma_c}{\ell}\ \theta^2 \qquad (72)$$

and therefore $\frac{dV}{dt} = 0.$

For V to be a Liapunov function, it has to fulfill the following conditions:

$$\begin{aligned}&(a)\ V(P,\theta) > 0 \text{ for } P>0 \text{ and } \theta>0\\ &(b)\ V(1,o) = 0 \\ &(c)\ \dot{V}(P,\theta) \le 0 \text{ for } P>0 \text{ and } \theta>0\end{aligned} \qquad (73)$$

If V of Equation (72) is modified as

$$V = (P - \text{In } P-1) - \frac{1}{2}\,\frac{\gamma_c}{\ell}\,\theta^2 \tag{74}$$

it is seen that Equation (71) is not altered, and for a negative reactivity input Equation (74) becomes

$$V = (P - \text{In } P-1) + \frac{1}{2}\,\frac{|\gamma|_c}{\ell}\,\theta^2 \tag{75}$$

The expression (P - In - P-1) > 0 for all values of P > 0. Then V satisfies condition (73a) for all values of P and $\theta > 0$, when γ is negative; for a positive value of γ, condition (73a) is satisfied whenever

$$(P - \text{In} - P-1) > \frac{1}{2}\,\frac{\gamma_c}{\ell}\,\theta^2$$

Condition (73b) is clearly satisfied for the steady state solution P = 1 and $\theta = 0$. Condition (73c) gives $\dot{V} = 0$ for both cases of positive and negative γ, which means that asymptotic stability has been ruled out. A similar result was found by Ergen and Weinberg (24) using the Hamiltonian approach. Due to its simplicity, this case has been added to this work to acquaint the reader with a general view of the necessary steps in the Liapunov's Method formulation. This case does not represent any meaningful practical situation.

Typical data for a thermal reactor is obtained from Solomon and Kastenberg (26), as shown in Table 1. From this data, ℓ is found to be 1.235×10^{-5} sec. An average heat capacity for liquid light water is used, 1 cal/cm^3- °C.

TABLE 1		
$\nu \Sigma_f$	0.49780	cm^{-1}
Σ_a	0.261	cm^{-1}
v	2.21×10^5	$cm\text{-}sec^{-1}$
β	0.0064	-
D	0.409718	cm
α	1.41	-
λ	0.080	sec^{-1}

The case of negative γ has been found to be stable for all values of P and θ, but the stability is only neutral ($\dot{V} = 0$).

The case of positive γ is represented in Figure 6, in which the system is seen to be unstable for most of the operating ranges of P and θ.

2. <u>One-Group Delayed Neutron, No Reactivity Input</u>

In this case β is included in the analysis and $\Delta k_o = 0$, the governing equations become

$$\ell\dot{P} = [(1-\beta)\Delta k-\beta]P + (1+\Delta k)\beta \tag{76}$$

$$c\dot{\theta} = P - 1 \tag{77}$$

$$\Delta k = \gamma\theta \tag{78}$$

Substitution of Equation (78) in Equation (77) yields

$$\ell\dot{P} = [(1-\beta)\gamma\theta-\beta]P + (1+\gamma\theta)\beta \tag{79}$$

"Cross multiplying" Equation (79) and Equation (77) gives, after some algebraic manipulations

$$\dot{P}- \frac{d}{dt} \text{In } P = \frac{c\gamma}{\ell} (1-\beta)\theta\dot{\theta}- \frac{\beta_c}{\ell} \dot{\theta} + \frac{\beta}{\ell} (1+\gamma\theta) \frac{(P-1)}{P} \tag{80}$$

Thus

$$\frac{d}{dt}[P-\text{In}P- \frac{1}{2} \frac{c}{\ell}(1-\beta)\gamma\theta^2+ \frac{\beta_c}{\ell} \theta] = \frac{\beta}{\ell} (1+\gamma\theta) \frac{(P-1)}{P} \tag{81}$$

Then

$$V = (P - \text{In } P-1) - \frac{1}{2} \frac{c\gamma}{\ell} (1-\beta)\theta^2 + \frac{\beta_c}{\ell} \theta \tag{82}$$

$$\dot{V} = \frac{\beta}{\ell} (1+\gamma\theta) \frac{(P-1)}{P} \tag{83}$$

It can be seen that the inclusion of the delayed neutrons complicates the Liapunov function, obtained by the same

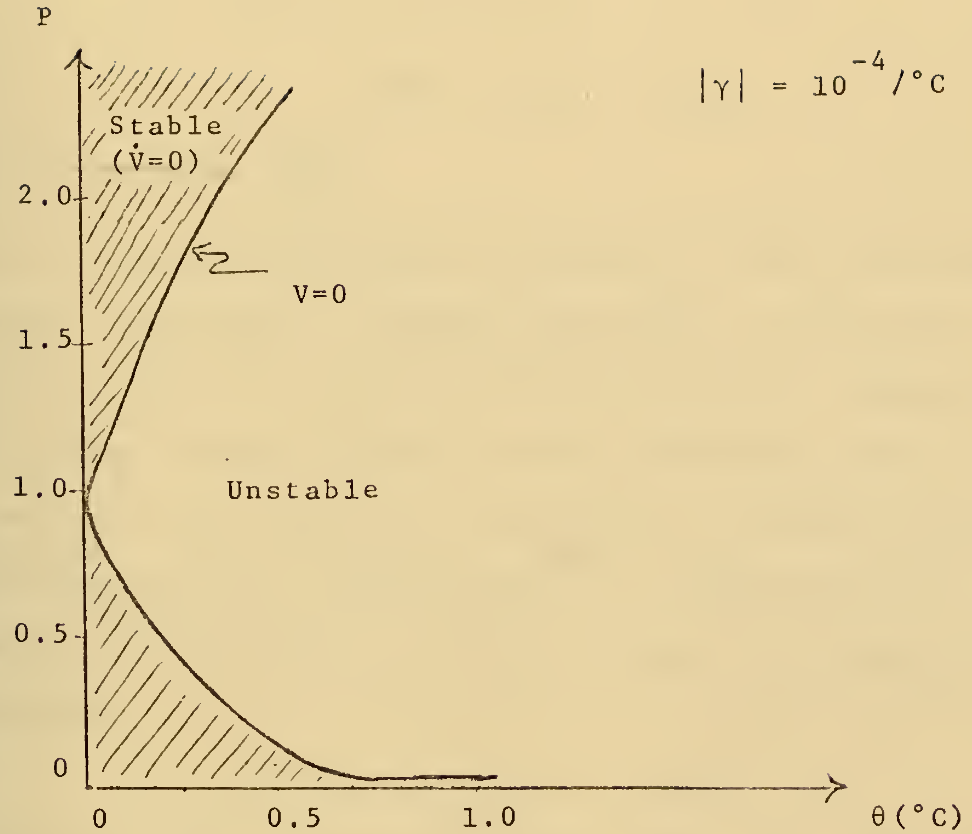

Figure 6. Stability domains for the case of no delayed neutrons, no Δk_o input, positive temperature coefficient of reactivity.

steps as for the previous case, but this case approaches a more real situation.

Two cases are analyzed:

a. Negative reactivity: $\gamma = -|\gamma|$

$$V = (P - \ln P - 1) + \frac{1}{2}\frac{c|\gamma|}{\ell}(1-\beta)\theta^2 + \frac{\beta_c}{\ell}\theta \tag{84}$$

$$\dot{V} = \frac{\beta}{\ell}(1-|\gamma|\theta)\frac{(P-1)}{P} \tag{85}$$

It is seen from Equation (84) that $V > 0$ for all positive values of P and θ. Normally $(1-|\gamma|\theta) > 0$, then for values of $0 < P \leq 1$, $\dot{V}$ is less or equal than zero. Clearly V(1.0) = 0, then V only vanishes at the equilibrium state. It is concluded that asymptotic stability is obtained whenever P be less than 1.0, and for all $\theta > 0$. Since $\Delta k_o = 0$, P cannot be larger than 1.0. Figure 7 shows the domain of stability.

b. Positive reactivity: $\gamma = |\gamma|$

Equations (82) and (83) represent this case. The condition of asymptotic stability is obtained whenever

$$(P - \ln P - 1) + \frac{\beta_c}{\ell}\theta > \frac{1}{2}\ \frac{c}{\ell}(1-\beta)\gamma\theta^2$$

and $\dot{P} < 1$.

This case presents a very narrow region of stability, as seen in Figure 8.

3. <u>One-Group Delayed Neutron, Step Reactivity Input</u>

The two previous cases have only theoretical interest because they do not represent a practical situation. The case of Δk_o input represents a real practical situation,

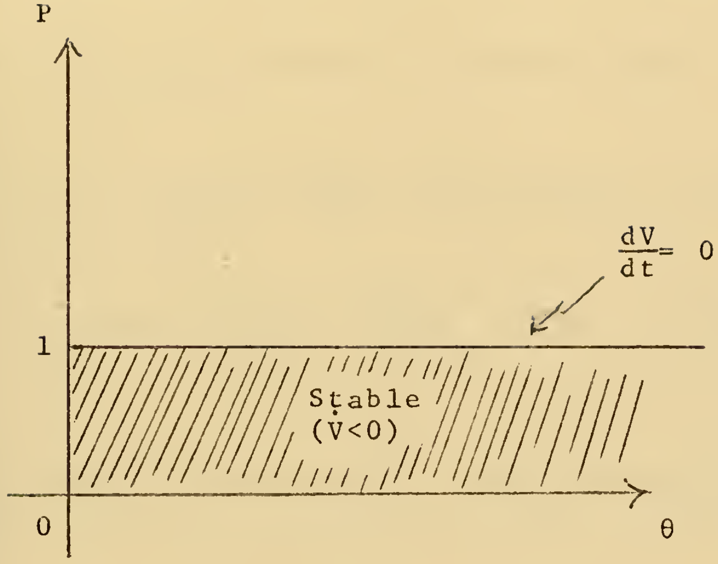

Figure 7. Stability domains for the case of delayed neutrons, no Δk_o input, negative temperature coefficient of reactivity.

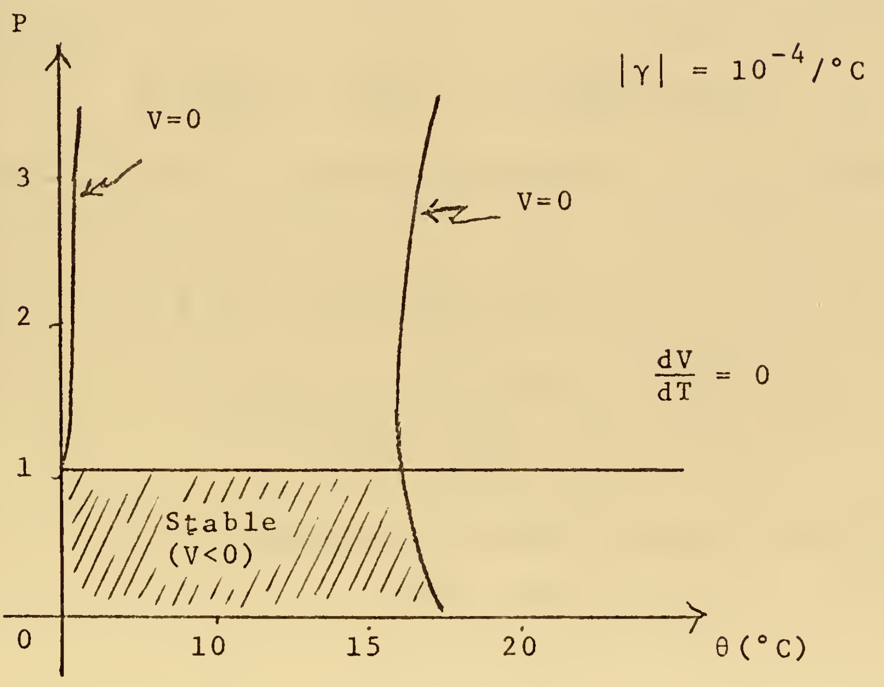

Figure 8. Stability domains for the case of delayed neutrons, no Δk_o input, positive temperature coefficient of reactivity.

i.e. start-up and shut-down, planned or unplanned motion of control rods. The governing equations for this system are:

$$\ell\dot{P} = [(1-\beta)(\Delta k_o + \gamma\theta)-\beta]\ P + (1+\Delta k_o + \gamma\theta)\beta \tag{86}$$

$$c\dot{\theta} = P - 1 \tag{87}$$

Substituting Equation (87) in Equation (86) yields, after simplifying terms:

$$\dot{P} - [\Delta k_o(1-\beta)-\beta]\ \frac{c}{\ell}\ \dot{\theta} = \frac{1}{\ell}(\Delta k_o + \beta\gamma\theta) + \frac{\gamma\theta}{\ell}\ (1-\beta)P \tag{88}$$

Then

$$\frac{d}{dt}\left[P-1-[\Delta k_o(1-\beta)-\beta]\ \frac{c}{\ell}\ \theta\right] = \frac{1}{\ell}(\Delta k_o+\beta\gamma\theta)+ \frac{\gamma}{\ell}(1-\beta)\theta P \tag{89}$$

Thus

$$V = (P-1) + [\beta- \Delta k_o(1-\beta)]\ \frac{c}{\ell}\ \theta \tag{90}$$

$$\dot{V} = \frac{1}{\ell}(\Delta k_o + \beta\gamma\theta) + \frac{\gamma}{\ell}\ (1-\beta)\theta P \tag{91}$$

For negative γ, Equation (91) remains unchanged and Equation (90) becomes

$$\dot{V} = \frac{1}{\ell}(\Delta k_o - \beta|\gamma|\ \theta) - \frac{|\gamma|}{\ell}(1-\beta)\theta P \tag{92}$$

In order to obtain a positive definite V, it is required that

$$P + [\beta - \Delta k_o(1-\beta)]\ \frac{c}{\ell}\ \theta > 1 \tag{93}$$

This condition is satisfied whenever

$$\Delta k_o < \frac{\beta}{1-\beta} \tag{94}$$

for all $\theta < 0$, in agreement with the linear theory. To get a negative definite $\dot{V}$, it is required that

$$\Delta k_o < [\beta + (1-\beta)P]\ |\gamma|\theta < 0 \tag{95}$$

This is the additional stability condition required by the

nonlinear theory. Equations (94) and (95) are specifying the bounds for asymptotic stability for a negative γ. It is noted that immediately after the step change in the multiplication factor, the neutron flux increases rapidly; this is referred to as the prompt jump, Lamarsh (25), and it is expressed as

$$P(t) \to \frac{\beta(1-\Delta k_o)}{\beta-\Delta k_o} P(o) \tag{96}$$

For positive γ, it is seen from Equation (91) that $\dot{V}$ will always be positive definite for all values of P and θ, representing an unstable situation. Data from Table 1 is used to specify domains of stability for various Δk_o inputs. The first bound for all these cases is provided by Equation (94) giving a $\Delta k_o < 1.00625\$$, in order to obtain a positive definite V, for all $P > 1$ and $\theta > 0$. It is clearly seen that P has to be greater than one, the steady state power, after a Δk_o insertion. The second bound is obtained, setting $\dot{V} = 0$ in Equation (92), Figures 9 and 10. After the insertion, the power will increase to an amount specified by Equation (96). From Figures 9 and 10, it can be seen that right after the Δk_o insertion, the system is unstable, and P and θ increase, until the boundary $\dot{V}=0$ is crossed, then the system becomes asymptotically stable and the trajectory returns to the equilibrium position, but when $\dot{V}=0$ is crossed again, the system turns to be unstable, and in that way, the trajectory keeps oscillating along the line $\dot{V}=0$. A similar situation is obtained by Meghreblian and

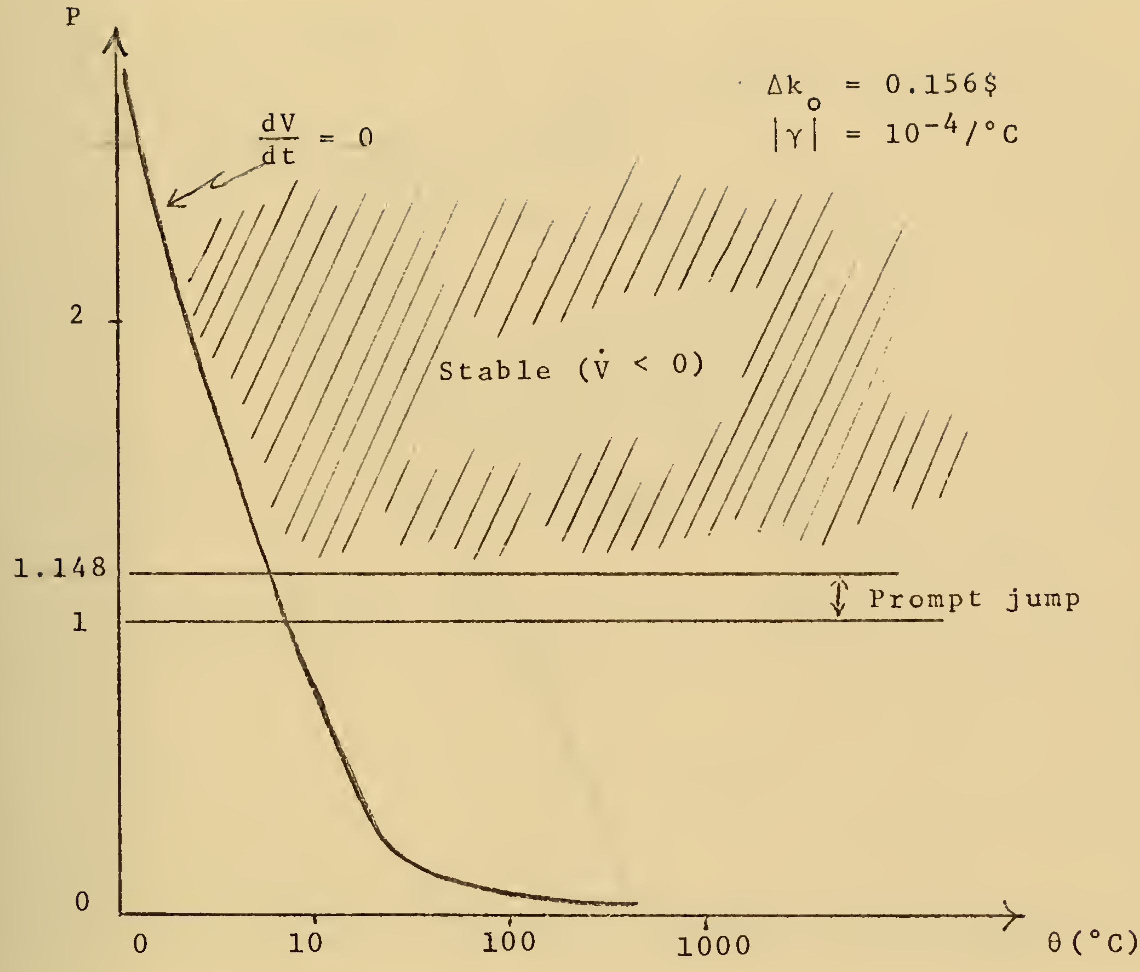

Figure 9. Stability domains for the case of delayed neutrons, step $\Delta k_o = 0.156\$$, negative temperature coefficient of reactivity.

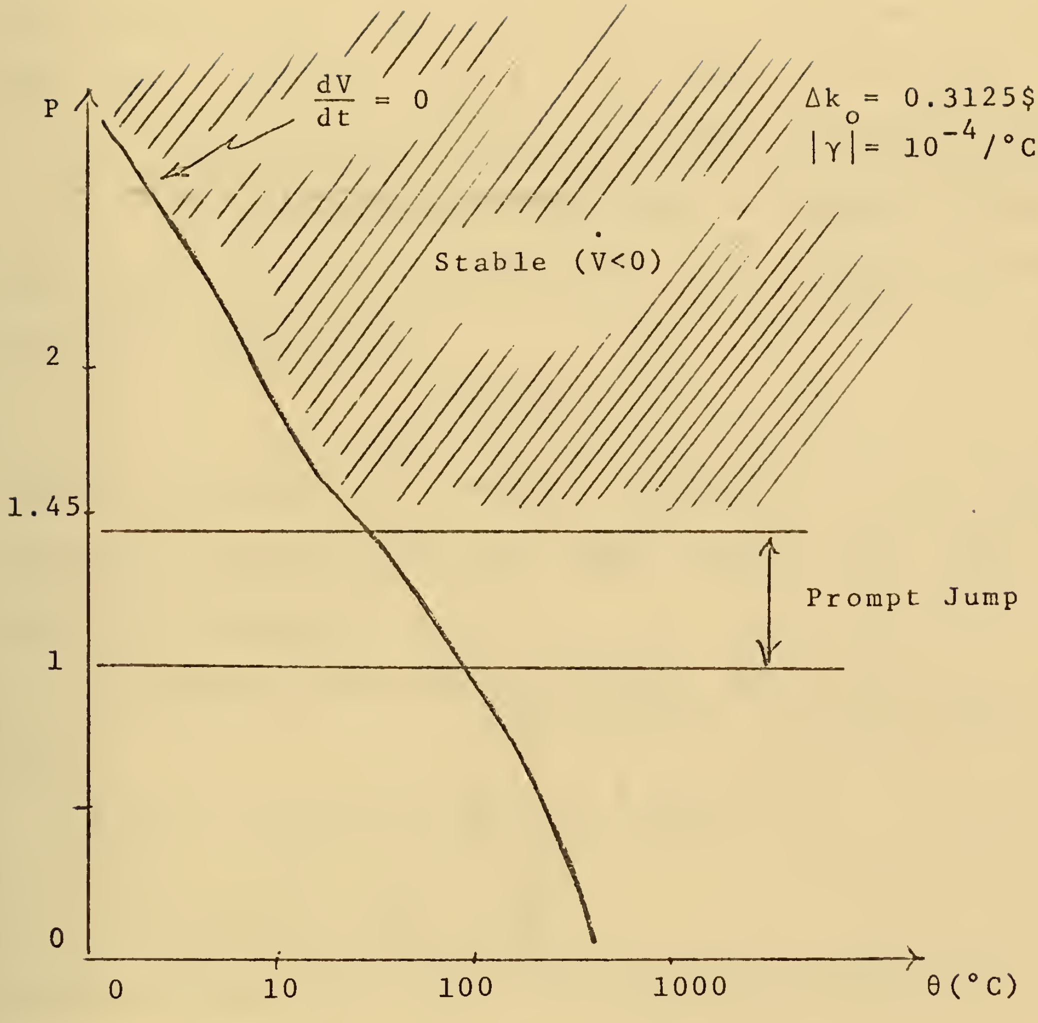

Figure 10. Stability domains for the case of delayed neutrons, step $\Delta k_o = 0.3125\$$, negative temperature coefficient of reactivity.

Holmes (22) when solving the system of differential equations. Their results showed that this is the case of damped oscillation, the presence of the delayed neutrons providing the damping force, thus, following an insertion of reactivity, the system will reach a new steady state power.

4. One-Group Delayed Neutron, Ramp Reactivity Input

The governing equations for the system are Equations (86) and (87), with the difference that now Δk_o is a function of t expressed as

$$\Delta k_o(t) = at \tag{97}$$

where a represents the rate of insertion of reactivity. Combining Equations (86) and (87) yields, after some algebraic manipulations

$$\ell \dot{P} -((1-\beta)\gamma c\ \theta\dot{\theta}+ \beta c\dot{\theta} -(1-\beta)atP - \beta at = \gamma\theta \tag{98}$$

Then

$$\frac{d}{dt}[P - 1 - \frac{1}{2}(1-\beta)\frac{\gamma_c}{\ell}\theta^2 + \frac{\beta_c}{\ell}\theta] =$$

$$= (1-\beta)\frac{at}{\ell}P + \frac{\gamma\theta}{\ell} + \beta\frac{at}{\ell} \tag{99}$$

From Equation (99), V and $\dot{V}$ are obtained to be:

$$V = P - 1 - \frac{1}{2}(1-\beta)\frac{\gamma c}{\ell}\theta^2 + \frac{\beta_c}{\ell}\theta \tag{100}$$

$$\dot{V} = (1-\beta)\frac{at}{\ell}P + \frac{\gamma\theta}{\ell} + \frac{\beta at}{\ell} \tag{101}$$

For a negative γ, Equations (100) and (101) become

$$V = P - 1 + \frac{\beta c}{\ell}\theta + \frac{1}{2}(1-\beta)\frac{\gamma c\theta^2}{\ell} \tag{102}$$

$$\dot{V} = (1-\beta)\frac{at}{\ell}P - \frac{\gamma\theta}{\ell} + \frac{\beta at}{\ell} \tag{103}$$

The following condition is obtained for asymptotic stability:

$$\Delta k_o(t) = at < \frac{\gamma\theta}{(1-\beta)P+\beta} \tag{104}$$

It can be seen that V is positive definite for all $P > 1$ and $\theta > 0$. Equation (104) is necessary to obtain a negative definite $\dot{V}$ to ensure asymptotic stability. For a positive γ, the system is inherently unstable. Equation (101) gives a positive definite $\dot{V}$ for all values of P and θ for $t \geq 0$. A typical ramp insertion of reactivity is shown in Figure A. During the reactivity insertion, the system is unstable, then $\dot{V} = 0$ is obtained after the ramp insertion has ended (10 sec) and the power and temperature have risen to an appropriate level. Setting $\dot{V} = 0$, in Equation (103), the following expression is obtained

$$P = \frac{|\gamma|\theta - \beta\Delta k_o}{(1-\beta)\Delta k_o} \tag{103a}$$

At the end of the Δk_o insertion ($t = 10$ sec), this expression implies that P depends linearly on θ, Figure 11.

5. <u>One-Group Delayed Neutron, General Reactivity Input</u>

The equations for this system are:

$$\ell\dot{P} = [(1-\beta)\Delta k-\beta]P + (1+\Delta k)\beta \tag{106}$$

$$c\dot{\theta} = P - 1 \tag{107}$$

$$\Delta k = \Delta k_o \frac{\gamma_D}{1-n} \left[T\left(\frac{300}{T}\right)^n - T_o\left(\frac{300}{T_o}\right)^n\right] \tag{108}$$

Let

$$R(T) = \frac{1}{1-n} \left[T\left(\frac{300}{T}\right)^n - T_o\left(\frac{300}{T_o}\right)^n\right] \tag{109}$$

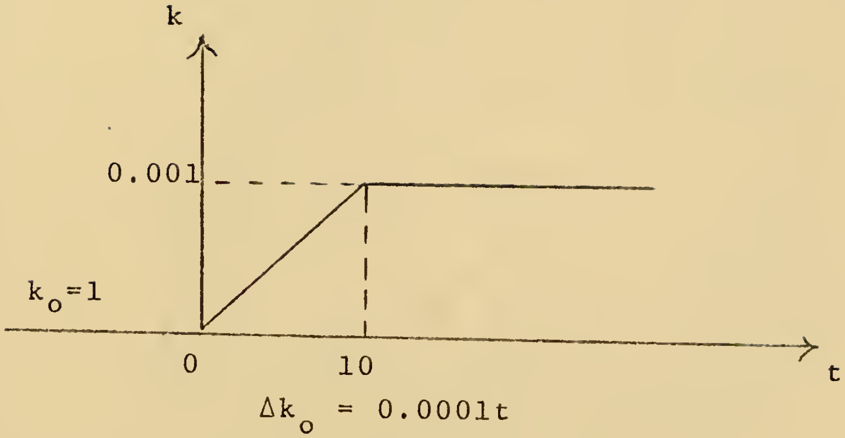

Figure A

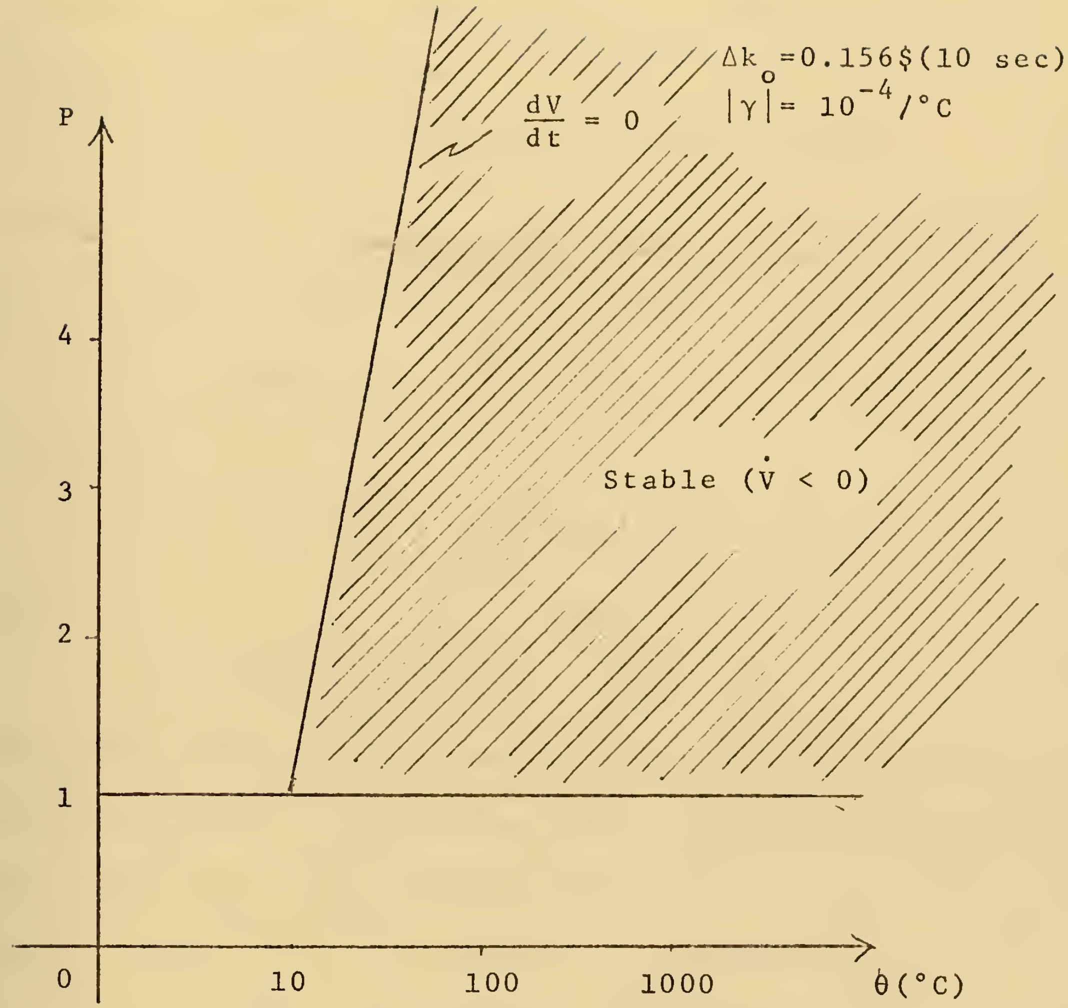

Figure 11. Stability domain at the end of a ramp insertion of reactivity.

Then Equation (106) becomes

$$\ell\dot{P} = [(1-\beta)(\Delta k_o+\gamma_D\cdot R(T)-\beta]P + [1+\gamma_D\cdot R(T)]\beta \qquad (110)$$

Combining Equations (110) and (107) yields, after some algebraic manipulations

$$\ell\dot{P} - (1-\beta)\Delta k_o c\dot{\theta}+\beta c\dot{\theta} = [(1-\beta)P+\beta]\gamma_D R+\Delta k_o \qquad (111)$$

Then

$$\frac{d}{dt}\left[P - 1 + \frac{\beta-(1-\beta)\Delta k_o}{\ell} c\theta\right] =$$

$$= [(1-\beta)P + \beta]\frac{\gamma_D\cdot R}{\ell} + \frac{\Delta k_o}{\ell} \qquad (112)$$

Therefore

$$V = P-1 + [\beta - (1-\beta)\ \Delta k_o]\ \frac{c\theta}{\ell} \qquad (113)$$

and

$$\dot{V} = [(1-\beta)P + \beta]\ \frac{\gamma_D\cdot R}{\ell} + \frac{\Delta k_o}{\ell} \qquad (114)$$

Equation (113) is independent of γ; therefore, V is positive definite whenever conditions (93) and (94) are satisfied. The similarity with the step insertion of reactivity can be seen. Equation (114) becomes, for negative γ

$$\dot{V} = \frac{\Delta k_o}{\ell} - [\beta + (1-\beta)P]\ \frac{|\gamma_D|\cdot R(T)}{\ell} \qquad (115)$$

Equation (115) is negative definite whenever

$$\Delta k_o < [\beta + (1-\beta)P]\ |\gamma_D|\cdot R(T) \qquad (116)$$

for all $n \neq 1$. The case $n = 1$ is the logarithmic case and Equation (116) becomes

$$\Delta k_o < 300[\beta+(1-\beta)P]\ |\gamma_D|\ \text{In}\ \frac{T}{T_o} \qquad (117)$$

The case of positive γ leads to an unstable situation due to the fact that Equation (114) is positive definite for all n, P and T. Data for this case are found in Thompson and Beckerley (23) and are typical of a fast reactor. The following specific case is analyzed:

(a) Oxide reactor with volume ratio UO_2 to P_uO_2 = 7

(b) Sodium density = 50%

(c) $\gamma_D = 1.28 \times 10^{-5}/°K$

(d) $n = 0.96$

(e) $\beta = 0.0033$

(f) $\Delta k_o = 0.156\$$

The domain of stability is shown in Figure 12, where the $\dot{V} = 0$ is obtained from Equation (115).

B. DISTRIBUTED-PARAMETER REACTOR SYSTEM

The reactor under consideration for this analysis has the following characteristics:

(a) fast reactor

(b) homogeneous, bare, slab reactor

(c) one-group delayed neutron

(d) Newton's law of cooling

(e) stationary-fuel reactor

and normal operating conditions (no accidents) are assumed.

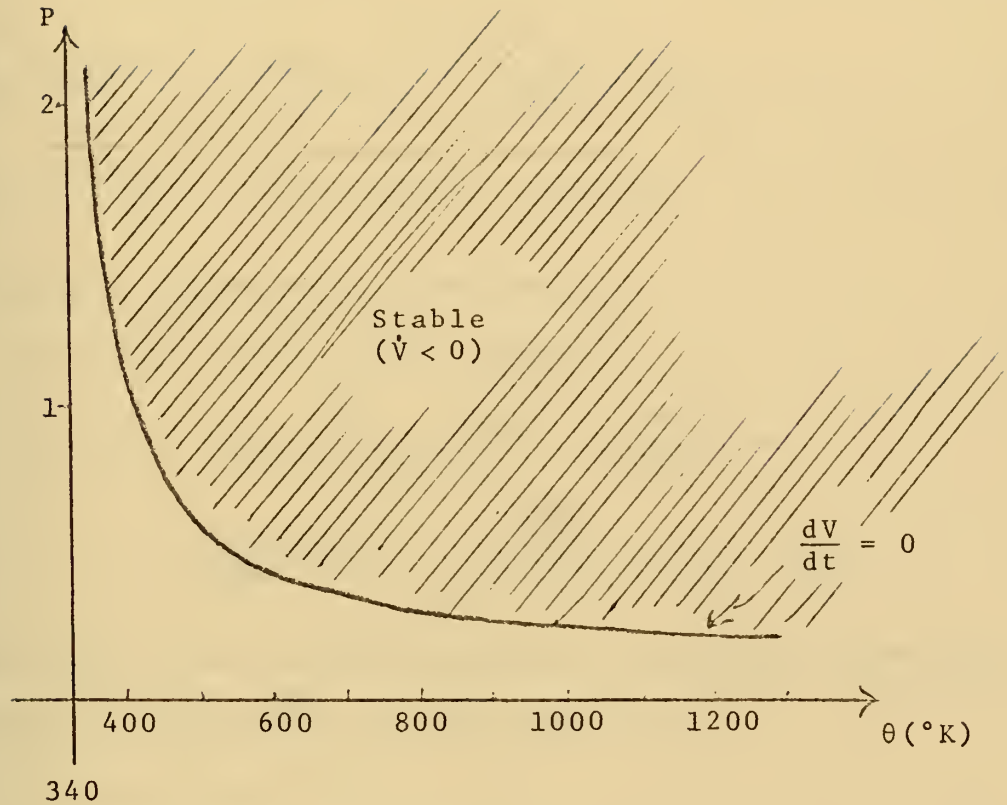

Figure 12. Stability domains for the case of delayed neutrons and general reactivity insertion.

The governing equations are:

$$\frac{\partial^2\psi(y,\tau)}{\partial y^2} + [(1-\beta)k\alpha-1]\psi(y,\tau) + \frac{k\alpha}{\nu\Sigma_f}\lambda\mathscr{C}(y,\tau) = \frac{\partial\psi(y,\tau)}{\partial\tau} \tag{37}$$

$$v\Sigma_a \frac{\partial\mathscr{C}(y,\tau)}{\partial\tau} = -\lambda\mathscr{C}(y,\tau) + \frac{\beta\alpha\ \Sigma_a k}{pg}\ \psi(y,\tau) \tag{46}$$

$$c_f v\Sigma_a \frac{\partial\theta(y,\tau)}{\partial\tau} = \varepsilon\ \Sigma_f\ \psi(y,\tau) - h\theta(y,\tau) \tag{57}$$

using the following feedback models:

(a) $k(y,\tau) = 1 + \Delta k_o + \gamma\theta(y,\tau)$ (63)

(b) $$k = 1+\Delta k_o + \frac{\gamma_D}{1-n}[T(\frac{a_1}{T})^n - T_o(\frac{a_1}{T_o})^n] + \frac{\gamma_E}{1+m}[T(\frac{T}{a_2})^m - T_o(\frac{T_o}{a_2})^m] + \gamma_o(T - T_o) \tag{66}$$

where Δk_o is an external positive reactivity insertion. The boundary conditions for the system are:

(a) $\psi(\pm W, \tau) = 0$

The initial conditions are:

(a) $\psi(y,o) = 0$

(b) $\mathscr{C}(y,o) = 0$

(c) $\theta(y,o) = 0$

The case n = 1 will give the logarithmic term in the doppler expression.

1. <u>One-Group Delayed Neutron, Step Reactivity Input</u>

To formulate the concept of stability a distance between the equilibrium position (ϕ_o, C_o, T_o) and any perturbed

position (ϕ, C, T) is defined. The variation of this norm, which is induced by the inner product of the Hilbert space in which the solutions of the system are defined, with time will provide information concerning the perturbed position.

Let

$$[d(\phi,\phi_o;C,C_o;T,T_o)]^2 = \int_{-W}^{W} (A_1^{\,2}\psi^2 + A_2^{\,2}\mathscr{C}^{\,2} + A_3^{\,2}\theta^2)\,dy \tag{118}$$

where

$$A_1^{\,2} = \frac{\varepsilon^2 \Sigma_f^{\,2}}{2} \tag{119}$$

$$A_2^{\,2} = \frac{\varepsilon^2 v^2 \Sigma_a^{\,2}}{2\beta^2 \nu^2} \tag{120}$$

$$A_3^{\,2} = C_f^{\,2} v^2 \Sigma_a^{\,2} \tag{121}$$

and

$$\psi = \phi - \phi_o$$

$$\mathscr{C} = C - C_o$$

$$\theta = T - T_o$$

The Liapunov function is then defined as

$$V(\psi, \mathscr{C}, \theta) = ||\bar{d}||^2 = < \bar{d}, \bar{d} > \tag{122}$$

which by definition is positive definite. It is noticed that the introduction of the constants A_1, A_2 and A_3 makes V represent the total rate of energy increase per unit volume of the system, which is defined as the distance between the equilibrium and the perturbed states.

V vanishes only at the equilibrium position (ϕ_o, C_o, T_o); therefore, the requirements of a negative definite $\dot{V}$ will provide the domain of asymptotic stability.

Thus

$$\frac{dV}{d\tau} = \int_{-W}^{W} \left(\varepsilon^2 \Sigma_f{}^2 \psi \frac{\partial \psi}{\partial y} + \frac{\varepsilon^2 v^2 \Sigma_a{}^2}{\beta^2 \nu^2} \mathscr{C} \frac{\partial \mathscr{C}}{\partial y} + 2C_f{}^2 v^2 \Sigma_a{}^2 \theta \frac{\partial \theta}{\partial y}\right) dy \tag{123}$$

Substituting Equations (37), (46) and (57) in Equation (123) yields

$$\frac{dV}{d\tau} = \varepsilon^2 \Sigma_f{}^2 \int_{-W}^{W} \Bigg\{ \psi \frac{\partial \psi}{\partial y^2} + [(1-\beta)k\alpha - 1]\psi^2 + \frac{k\lambda\alpha}{\nu\Sigma_f} \mathscr{C} \psi +$$

$$+ \frac{v\Sigma_a{}^2 k\alpha}{\beta\nu^2 \Sigma_f{}^2} \mathscr{C} \psi - \frac{v\Sigma_a \lambda}{\beta^2 \nu^2 \Sigma_f{}^2} \mathscr{C}^2 + \frac{2C_f \Sigma_a{}^2 k\alpha}{\nu\varepsilon\Sigma_f{}^2} \theta\psi - \frac{2C_f v\Sigma_a h}{\varepsilon^2 \Sigma_f{}^2} \theta^2 \quad dy \Bigg\} \tag{124}$$

Integration by parts of the first term and using the boundary condition $\psi(\pm W, \tau) = 0$ yields

$$\int_{-W}^{W} \psi \frac{\partial^2 \psi}{\partial y^2} dy = - \int_{-W}^{W} \left(\frac{\partial \psi}{\partial y}\right)^2 dy \tag{125}$$

Knops and Wilkes (27), in their work, provide the inequalities useful for this kind of analysis. The so-called "eigenvalue inequality" plays an important role in this specific problem. For a system defined by the eigenvalue problem

$$\nabla^2 \mu + \lambda \mu = 0$$

in domain with $\mu(a,t) = \mu(-a,t) = 0$, on the boundary, the following inequality can be stated:

$$\lambda \int_{-a}^{a} \mu^2 \, dy \leq \int_{-a}^{a} \left(\frac{\partial \mu}{\partial y}\right)^2 dy \tag{126}$$

an inequality that can be proved using the calculus of variations. (See Appendix A). In applying this inequality to this work, it is assumed that the perturbed reactor has the fundamental eigenvalue not appreciably different from that of the unperturbed reactor, then

$$\lambda = \frac{\pi^2}{4W^2} = B^2L^2 \tag{127}$$

Thus, Equation (126) becomes

$$\int_{-W}^{W} \left(\frac{\partial\psi}{\partial y}\right)^2 dy \geq B^2L^2 \int_{-W}^{W} \psi^2 \, dy \tag{128}$$

Substitution of this result in Equation (124) yields

$$\begin{aligned} \frac{dV}{dt} \leq - \varepsilon^2\Sigma_f \int_{-W}^{W} \Bigg\{ & B^2L^2\psi^2 - [(1-\beta)k\alpha-1]\psi^2 - \\ & - \frac{k\lambda\alpha}{\nu\Sigma_f} \mathscr{C}\psi - \frac{\nu\Sigma_a{}^2 k\alpha}{\beta\nu^2\Sigma_f{}^2} \mathscr{C}\psi + \frac{\nu\Sigma_a\lambda}{\beta^2\nu^2\Sigma_f{}^2} \mathscr{C}^2 - \\ & - \frac{2C_f\nu\Sigma_a{}^2 k\alpha}{\nu\varepsilon\Sigma_f{}^2} \theta\psi + \frac{2C_f\nu\Sigma_a h}{\varepsilon^2\Sigma_f{}^2} \theta^2 \Bigg\} dy \end{aligned} \tag{129}$$

After introduction of the feedback term, the following expression is finally obtained:

$$\frac{dV}{d\tau} \leq - \varepsilon^2 \Sigma_f^{\,2} \alpha \int_{-W}^{W} \Bigg\{ [1-(1-\beta)(1+\Delta k_o)]\, \psi^2$$

$$+ (1-\beta)\gamma\theta\psi^2 - (1+\Delta k_o)\left(\frac{\lambda}{\nu\Sigma_f} + \frac{v\Sigma_a^{\,2}}{\beta\nu^2\Sigma_f^{\,2}}\right)\mathscr{E}\psi +$$

$$+ \left(\frac{\lambda}{\nu\Sigma_f} + \frac{v\Sigma_a^{\,2}}{\beta\nu^2\Sigma_f^{\,2}}\right)\gamma\theta\,\mathscr{E}\psi + \frac{v\Sigma_a\lambda}{\beta^2\nu^2\Sigma_f^{\,2}\alpha}\,\mathscr{E}^2 \quad +$$

$$+ \frac{2C_f v\Sigma_a h}{\varepsilon^2\Sigma_f^{\,2}\alpha}\,\theta^2 - \frac{2C_f v\Sigma_a^{\,2}}{\varepsilon\Sigma_f^{\,2}\nu}(1+\Delta k_o)\,\theta\psi +$$

$$+ \frac{2C_f v\Sigma_a^{\,2}}{\varepsilon\Sigma_f^{\,2}\nu}\,\gamma\theta^2\psi \Bigg\}\, dy \qquad (130)$$

Let

$$a_1 = 1 - (1-\beta)(1+\Delta k_o) \qquad (131)$$

$$a_2 = \frac{v\Sigma_a\lambda}{\beta^2\nu^2\Sigma_f^{\,2}\alpha} \qquad (132)$$

$$a_3 = \frac{2C_f v\Sigma_a h}{\varepsilon^2\Sigma_f^{\,2}\alpha} \qquad (133)$$

$$a_4 = \frac{\lambda}{\nu\Sigma_f} + \frac{v\Sigma_a^{\,2}}{\beta\nu^2\Sigma_f^{\,2}} \qquad (134)$$

$$a_5 = \frac{2C_f v\Sigma_a^{\,2}}{\varepsilon\Sigma_f^{\,2}\nu} \qquad (135)$$

Then Equation (130) can be expressed as

$$\frac{dV}{d\tau} \leq - \varepsilon^2 \Sigma_f{}^2 \alpha \left\{ \int_{-W}^{W} [a_1 \psi^2 + a_2 \mathcal{E}^2 + a_3 \theta^2 + \right.$$

$$+ (1-\beta) \gamma\theta\psi^2 - (1+\Delta k_o) a_4 \mathcal{E}\psi + a_4 \gamma\theta\mathcal{E}\psi -$$

$$\left. - (1+\Delta k_o) a_5 \theta\psi + a_5 \gamma\theta^2\psi] \, dy \right\} \qquad (136)$$

To obtain asymptotic stability, it is necessary to prove that the expression inside the curly brackets is positive definite. A strategy successfully employed by Buis and Vogt (28) will be used here. Equation (136) can be written as follows:

$$\frac{dV}{d\tau} \leq - \varepsilon^2 \Sigma_f{}^2 \alpha \left\{ [\int_{-W}^{W} (a_1 \psi^2 + a_2 \mathcal{E}^2 + a_3 \theta^2) dy] \cdot \right.$$

$$\left. \cdot \left[1 - \frac{\int_{-W}^{W} [(1+\Delta k_o) a_4 \mathcal{E}\psi - a_4 \gamma\theta\mathcal{E}\psi - (1-\beta)\gamma\theta\psi^2 + (1+\Delta k_o) a_5 \theta\psi - a_5 \gamma\theta^2\psi] dy}{\int_{-W}^{W} (a_1 \psi^2 + a_2 \mathcal{E}^2 + a_3 \theta^2) dy} \right] \right\} \qquad (137)$$

Equation (137) can be expressed concisely as

$$\frac{dV}{d\tau} \leq - R \cdot P \cdot Q \qquad (138)$$

where $R = \varepsilon^2 \Sigma_f{}^2 \alpha$ is essentially a positive constant,

$P = \int_{-W}^{W} (a_1 \psi^2 + a_2 \mathcal{E}^2 + a_3 \theta^2)$ is greater than zero whenever a_1 a_2 and a_3 are greater than zero. It is already known that a_2 and a_3 are positive constants.

Thus $a_1 = 1 - (1-\beta)(1+\Delta k_o)$ has to be greater than zero in order to have $P > 0$.

From this condition, the first bound for stability is obtained

$$\boxed{\Delta k_o < \frac{\beta}{1-\beta}} \tag{139}$$

in agreement with the linear theory. To prove that Q, second expression in brackets in Equation (137), is positive definite, the Buniakowsky-Schwarz inequality will repeatedly be used

$$\int_{-a}^{a} fg \, dy \leq \left(\int_{-a}^{a} |f|^2 dy \right)^{1/2} \left(\int_{-a}^{a} |g|^2 dy \right)^{1/2} \tag{140}$$

Due to the positive Δk_o insertion, the system's state variables ψ, $\mathscr{C}$, θ are positive for any $t > 0$; therefore, the absolute value in the inequality is not necessary. It is also known that

$$\int_{-W}^{W} A_1^2 \, \psi^2 \, dy \leq \|\bar{d}\|^2 \tag{141}$$

$$\int_{-W}^{W} A_2^2 \mathscr{C}^2 \, dy \leq \|\bar{d}\|^2 \tag{142}$$

$$\int_{-W}^{W} A_3^2 \, \theta^2 \, dy \leq \|\bar{d}\|^2 \tag{143}$$

State variables in engineering have the properties of functions in the Hilbert space, and the norm in this space is specifically defined as

$$\left(\int_{-a}^{a} \mu^2 dx \right)^{1/2} = \|\mu\|_{L_2} \tag{144}$$

Due to the fact that the state variables in the present work, ψ, $\mathscr{C}$ and θ, are continuous in space and time domains together with their derivatives to arbitrary kth order, it

is expected that they also have the properties of functions in the Banach space. The norm in the Banach space (L_m) can be defined as:

$$\left(\int_{-a}^{a} \mu^m \, dx \right)^{1/m} = \|\mu\|_{L_m} \quad \text{for } m > 1 \tag{144a}$$

Then, repeated applications of these inequalities yields

$$1 - \frac{\frac{(1+\Delta k_o)}{A_1}\left(\frac{a_4}{A_2} + \frac{a_5}{A_3}\right)\|\bar{d}\|^2 - \frac{2W\gamma}{A_1}\left(\frac{a_4}{A_2 A_3} + \frac{(1-\beta)}{A_1 A_3} + \frac{a_5}{A_3^2}\right)\|\bar{d}\|^3}{\min\left(\frac{a_1}{A_1^2}, \frac{a_2}{A_2^2}, \frac{a_3}{A_3^2}\right)\|\bar{d}\|^2} > 0 \tag{145}$$

where the minimum is taken in order to assure the inequality. Therefore, for Q > 0, it is necessary that

$$\|\bar{d}\| \geq \frac{\frac{(1+\Delta k_o)}{A_1}\left(\frac{a_4}{A_2} + \frac{a_5}{A_3}\right) - \min\left(\frac{a_1}{A_1^2}, \frac{a_2}{A_2^2}, \frac{a_3}{A_3^2}\right)}{\frac{2W\gamma}{A_1}\left(\frac{a_4}{A_2 A_3} + \frac{(1-\beta)}{A_1 A_3} + \frac{a_5}{A_3^2}\right)} \tag{146}$$

This is the additional stability condition prescribed by the nonlinear theory. The inequality (146) gives the second bound for asymptotic stability of the system, which represents a spherical surface in the first octant, i.e. ψ, $\mathcal{E}$ and $\theta \geq 0$. Domains of stability are represented in Figure 13. The solutions for $\mathcal{E}(y,\tau)$ and $\theta(y,\tau)$ can be obtained from Equations (46) and (57), as follows:

$$\mathcal{E}(y,\tau) = \int_{o}^{\tau} H_1(\tau-\tau')\, \psi(y,\tau')\, d\tau' \tag{147}$$

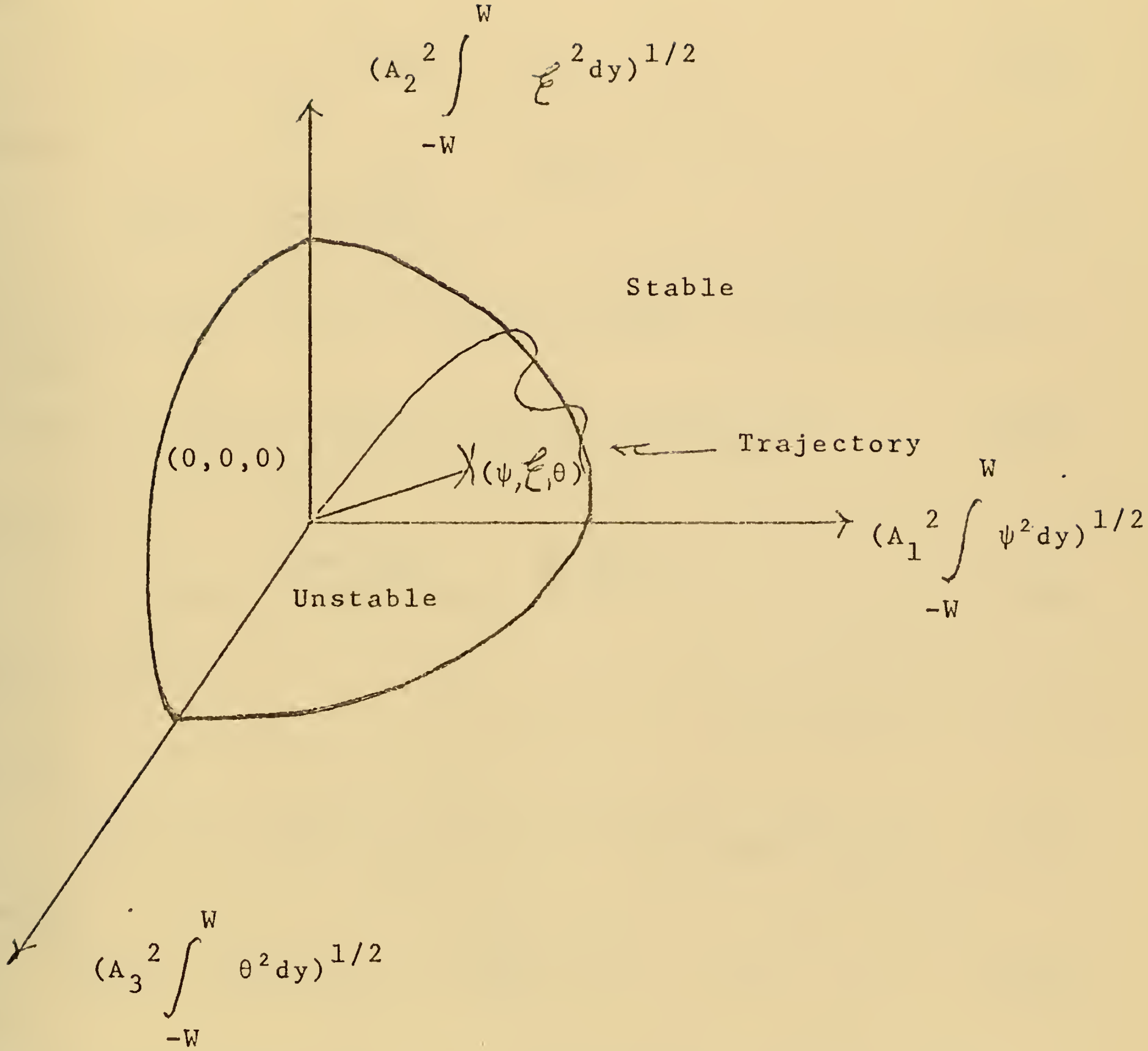

Figure 13. Schematic of the spherical surface determining stability domains for the distributed parameter reactor system after a Δk_o insertion.

where

$$H_1(\tau-\tau') = \nu\Sigma_f\beta\, e^{-\frac{\lambda}{v\Sigma_a}(\tau-\tau')} \tag{148}$$

and

$$\theta(y,\tau) = \int_0^{\tau} H_2(\tau-\tau')\ \psi(y,\tau')\ d\tau' \tag{149}$$

where

$$H_2(\tau-\tau') = \frac{\varepsilon\Sigma_f}{Cv\Sigma_a}\ e^{-\frac{h}{Cv\Sigma_a}(\tau-\tau')} \tag{150}$$

Application of the mean value theorem, in the time domain, yields

$$\mathscr{E}(y,\tau) = \psi(y,\bar{\tau})\int_0^{\tau} H_1(\tau-\tau')d\tau' = \psi(y,\bar{\tau})T_1(\tau) \tag{151}$$

where

$$T_1(\tau) = \frac{\beta\nu v\Sigma_f\Sigma_a}{\lambda}\left(1 - e^{-\frac{\lambda}{v\Sigma_a}\tau}\right) \tag{152}$$

and $0 < \bar{\tau} < \tau$.

Also:

$$\theta(y,\tau) = \psi(y,\bar{\tau})\int_0^{\tau} H_2(\tau-\tau')\ d\tau' = \psi(y,\bar{\tau})T_2(\tau) \tag{153}$$

where

$$T_2(\tau) = \frac{\varepsilon\Sigma_f}{h}\left(1 - e^{-\frac{h}{cv\Sigma_a}\tau}\right) \tag{154}$$

and $0 < \bar{\tau} < \tau$.

Substituting Equations (151) and (153) in Equation (118) results in

$$\|\bar{d}\|^2 = \int_{-W}^{W} A_1{}^2\psi^2(y,\tau)dy + \int_{-W}^{W} A_2{}^2\psi 2(y,\bar{\tau})T_1{}^2(\tau)dy + \int_{-W}^{W} A_3{}^2\psi^2(y,\bar{\tau})\ T_2{}^2(\tau)\ dy \tag{155}$$

Applying the mean value theorem, in the space domain, yields:

$$\|\bar{d}\|^2 = A_1{}^2\psi^2(\bar{y},\bar{\tau}) + 2WA_2{}^2\psi^2(\bar{y},\bar{\tau})\ T_1{}^2(\tau)$$

$$+ 2WA_3{}^2\psi^2(\bar{y},\bar{\tau})\ T_2{}^2(\tau) \tag{156}$$

or

$$\|\bar{d}\| = \psi(\bar{y},\bar{\tau})\ \sqrt{A_1{}^2 + 2WA_2{}^2T_1{}^2(\tau) + 2WA_3{}^2T_2{}^2(\tau)} \tag{157}$$

where $0 < \bar{\tau} < \tau$ and $-W < \bar{y} < W$.

Then the mean value flux can be expressed explicitly as follows:

$$\psi(\bar{y},\bar{\tau}) = \frac{\|\bar{d}\|}{\sqrt{A_1{}^2 + 2WA_2{}^2T_1{}^2(\tau) + 2WA_3{}^2T_2{}^2(\tau)}} \tag{158}$$

Typical values for the fast reactor nuclear parameters are listed in Table 2, after Solomon and Kastenberg (26). Using these parameters, the constants for the system are found to be:

$$a_2 = 3.292 \times 10^{13}\ \text{cm}^2\text{-sec}^{-2}$$

$$a_3 = 6.185 \times 10^{27}\ \text{cm}^{-2}\text{-}^\circ\text{C}^{-1}\text{-sec}^{-1}$$

$$a_4 = 9.050 \times 10^{9}\ \text{cm-sec}^{-1}$$

$$a_5 = 1.470 \times 10^{16}\ \text{cm}^{-2}\text{-}^\circ\text{C}^{-1}\text{-sec}^{-1}$$

$$A_1{}^2 = 2.932 \times 10^{-26}\ \text{cal}^2\text{-cm}^{-2}$$

$$A_2{}^2 = 2.780 \times 10^{-8}\ \text{cal}^2\text{-sec}^{-2}$$

$$A_3{}^2 = 1.920 \times 10^{2}\ \text{cal}^2\text{-cm}^{-6}\text{-}^\circ\text{C}^{-1}\text{-sec}^{-2}$$

$$W = 3.0$$

and

$$T_1(\tau) = 70.05\ (1 - e^{-0.0000314\tau}) \tag{159}$$

$$T_2(\tau) = 2.38 \times 10^{-12}(1 - e^{-0.0000448\tau}) \tag{160}$$

From Equation (139), the first bound is given by:

$$\Delta k_o < 1.00331\$$$

From Equation (146), the norm, second bound, is obtained to be:

$$\|\bar{d}\| \geq 2 \times 10^5$$

for a $\Delta k_o = 0.156\$$. The mean value flux from Equation (158) is shown in Figure 14. The physical meaning of this mean value flux $\psi(\bar{y},\bar{\tau})$ is depicted in Figure 15. The mean value flux provides an order of magnitude information of the increase in neutron population at the time the surface of stability is reached. From Figure 14, it is seen that a fast rising flux will quickly reach the surface, whereas a more slowly-rising flux will take longer to reach the stability domain.

2. <u>One-Group Delayed Neutron, Space Dependent Step Reactivity Input</u>

This system is a special case of the previously analyzed system. It is desired to find an answer to the following question: Given a positive reactivity insertion Δk_o, is it safer to insert this reactivity uniformly across the reactor, Figure 16a, or to insert less reactivity in the central region, Figure 16b, provided the total reactivity insertion is the same?

Equation (129) can be rewritten as:

$$\frac{dV}{d\tau} \leq -\varepsilon^2\Sigma_f{}^2\alpha \int_{-W}^{W} \Bigg\{ [1-(1-\beta)k]\psi^2 + \frac{\nu\Sigma_a\lambda}{\beta^2\nu^2\Sigma_f{}^2\alpha} \mathscr{C}^2 +$$

$$+ \frac{2C_f\nu\Sigma_a h}{\varepsilon^2\Sigma_f{}^2\alpha}\theta^2 - \frac{k}{\nu\Sigma_f}\left(\lambda + \frac{\nu\Sigma_a{}^2}{\beta\nu\Sigma_f}\right)\mathscr{C}\psi - \frac{2C_f\nu\Sigma_a{}^2 k}{\nu\varepsilon\Sigma_f{}^2}\theta\psi \Bigg\} dy \qquad (161)$$

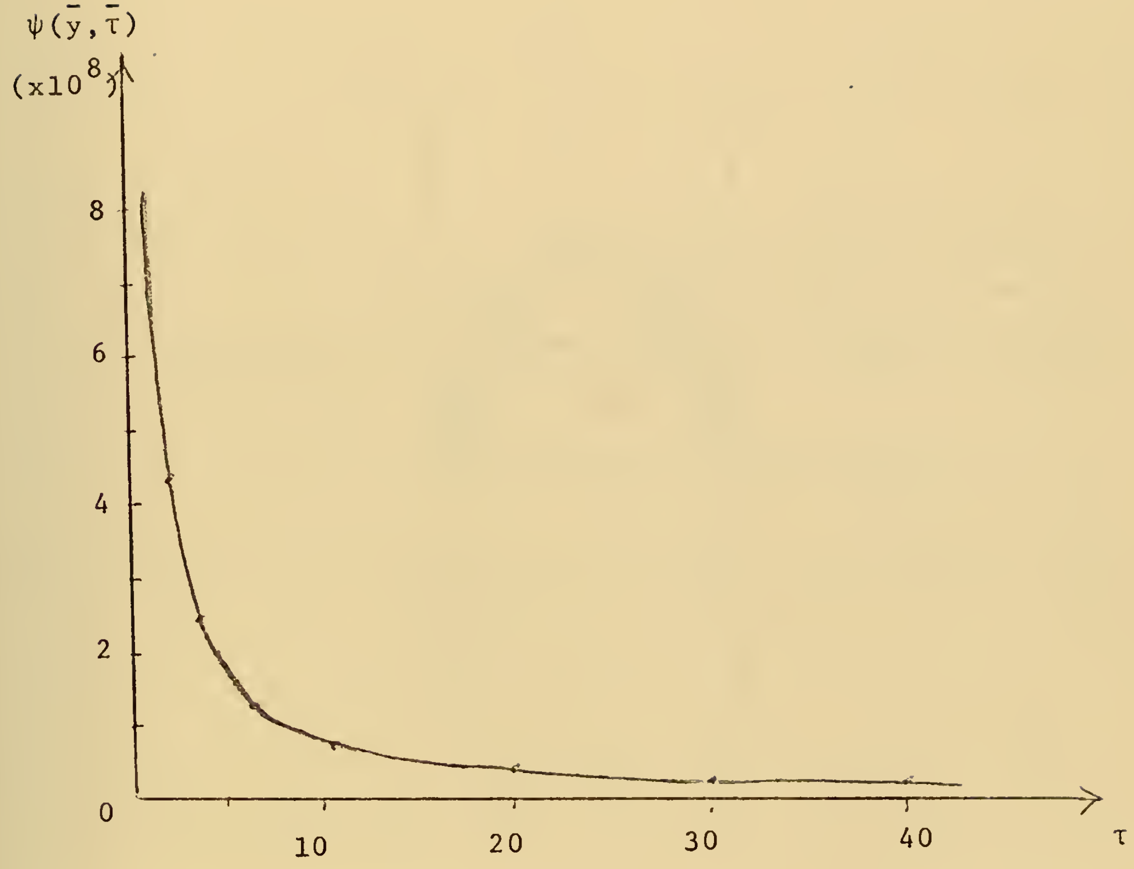

Figure 14. Mean value flux vs. dimensionless time plot.

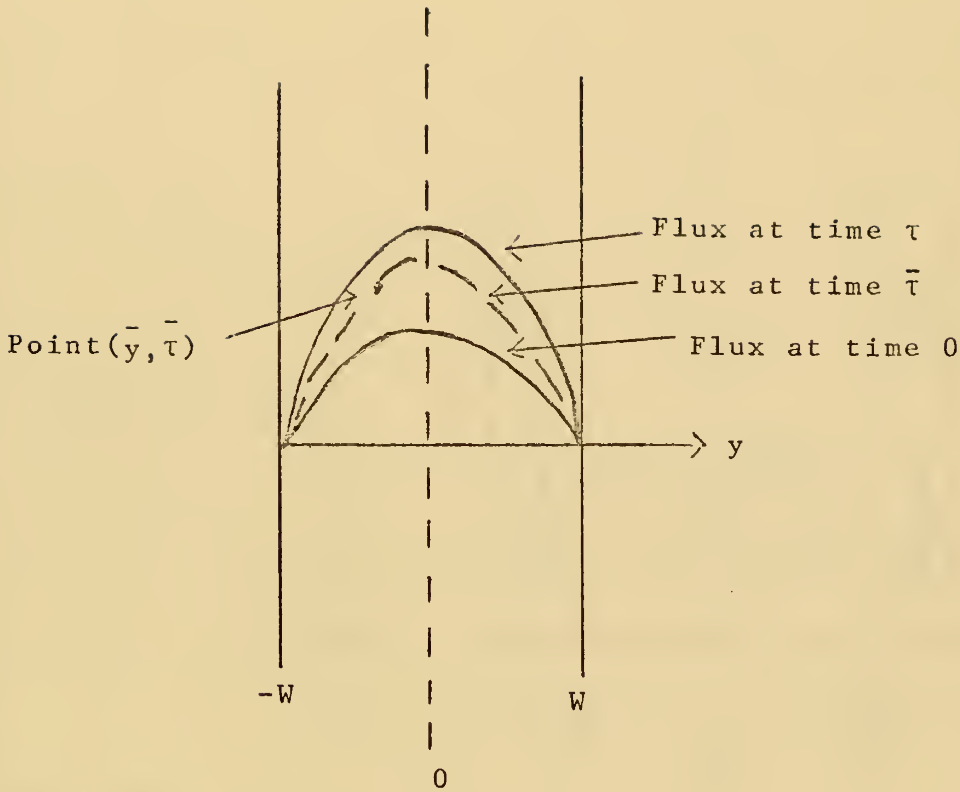

Figure 15. Mean value flux schematic in the slab reactor.

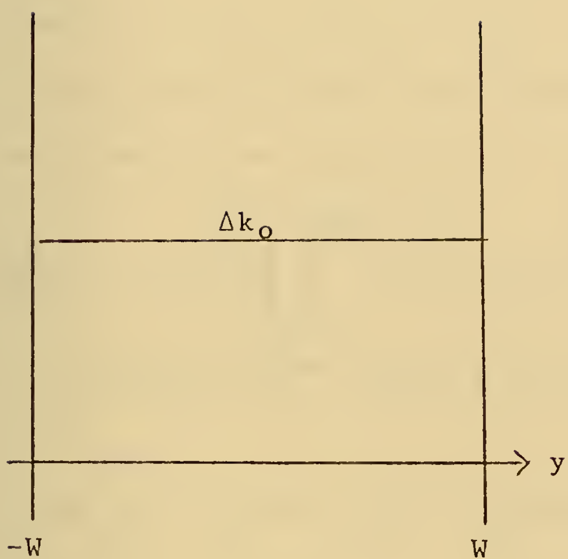

Figure 16a

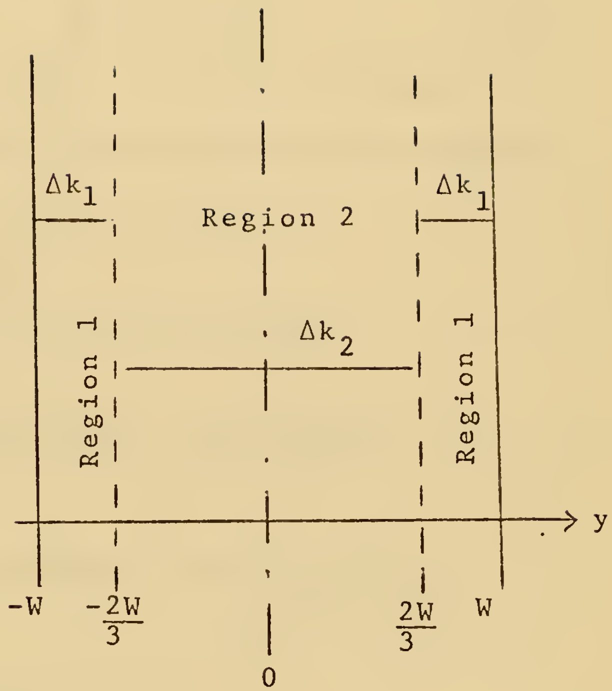

Figure 16b

Figure 16. Space-dependent Δk_o insertion.

Equation (161) can be rewritten as:

$$\frac{dV}{d\tau} \leq -\varepsilon^2\Sigma_f\alpha \left[\int_{-W}^{-2W/3} \{-\}\, dy + \int_{-2W/3}^{2W/3} \{-\}\, dy + \int_{2W/3}^{W} \{-\}\, dy \right. \tag{162}$$

Three steps Δk_o are introduced in the following way:

$$k = 1 + \Delta k_1 - \gamma\theta \qquad \text{for} \qquad -W \leq y \leq -\frac{2W}{3} \tag{163}$$

$$k = 1 + \Delta k_2 - \gamma\theta \qquad \text{for} \qquad -\frac{2W}{3} \leq y \leq \frac{2W}{3} \tag{164}$$

$$k = 1 + \Delta k_1 - \gamma\theta \qquad \text{for} \qquad \frac{2W}{3} \leq y \leq W \tag{165}$$

Substituting Equations (163), (164) and (165) in Equation (163) yields, after some algebraic manipulations and introduction of the constants a_1, a_2, a_3, a_4 and a_5:

$$\begin{aligned}
\frac{dV}{d\tau} \leq - \varepsilon^2\Sigma_f{}^2\alpha \Bigg\{ & \int_{-W}^{-2W/3} [a_1^{1}\psi^2 + a_2\xi^2 + a_3\theta^2 + (1-\beta)\gamma\theta\psi^2 - \\
& -(1+\Delta k_1)a_4\xi\psi + a_4\gamma\theta\xi\psi - (1+\Delta k_1)a_5\theta\psi + a_5\gamma\theta^2\psi]dy + \\
& + \int_{-2W/3}^{2W/3} [a_1''\psi^2 + a_2\xi^2 + a_3\theta^2 + (1-\beta)\gamma\theta\psi^2 - (1+\Delta k_2)a_4\xi\psi + \\
& + a_4\gamma\theta\xi\psi - (1+\Delta k_2)a_5\theta\psi + a_5\gamma\theta^2\psi]\, dy + \\
& + \int_{2W/3}^{W} [a_1^{1}\psi^2 + a_2\xi^2 + a_3\,\theta^2 + (1-\beta)\gamma\theta\psi^2 - (1+\Delta k_1)a_4\xi\psi + \\
& + a_4\gamma\theta\xi\psi - (1+\Delta k_1)a_5\,\theta\psi + a_5\,\gamma\theta^2\psi]dy \Bigg\}
\end{aligned} \tag{166}$$

where

$$a_1^{1} = 1 - (1-\beta)(1+\Delta k_1) \tag{167}$$

$$a_1'' = 1 - (1-\beta)(1+\Delta k_2) \tag{168}$$

Following the same steps as in the previous analysis, the following conditions for asymptotic stability are obtained:

$$\Delta k_1 < \frac{\beta}{1-\beta} \tag{169a}$$

$$\Delta k_2 < \frac{\beta}{1-\beta} \tag{169b}$$

conditions which agree with the linear theory, and

$$\|\bar{d}\|_1 \geq \frac{\frac{(1+\Delta k_1)}{A_1}\left(\frac{a_4}{A_2} + \frac{a_5}{A_3}\right) - \min\left(\frac{a_1}{A_1^{\ 2}}, \frac{a_2}{A_2^{\ 2}}, \frac{a_3}{A_3^{\ 2}}\right)}{\frac{W}{3}\frac{\gamma}{A_1}\left(\frac{a_4}{A_2 A_3} + \frac{(1-\beta)}{A_1 A_3} + \frac{a_5}{A_3^{\ 2}}\right)} \tag{170}$$

$$\|\bar{d}\|_2 \geq \frac{\frac{(1+\Delta k_2)}{A_1}\left(\frac{a_4}{A_2} + \frac{a_5}{A_3}\right) - \min\left(\frac{a_1}{A_1^{\ 2}}, \frac{a_2}{A_2^{\ 2}}, \frac{a_3}{A_3^{\ 2}}\right)}{\frac{4W}{3}\frac{\gamma}{A_1}\left(\frac{a_4}{A_2 A_3} + \frac{(1-\beta)}{A_1 A_3} + \frac{a_5}{A_3^{\ 2}}\right)} \tag{171}$$

These are the additional condition of stability required by the nonlinear theory. In order for the total reactivity insertion to remain the same for both strategies, one must have

$$\frac{1}{2W}\int_{W}^{-W} \Delta k_o dy = \frac{6}{W}\int_{-W}^{-2W/3} \Delta k_1 dy + \frac{3}{4W}\int_{-2W/3}^{2W/3} \Delta k_2 dy \tag{172}$$

From Figure 17, it is seen that in Region 2 in which $\Delta k_2 < \Delta k_o$, $\|\bar{d}\|_2 > \|\bar{d}\|$, so that the region of stability is attained sooner than in the case of uniform Δk_o insertion; but $\|\bar{d}\|_1 > \|\bar{d}\|$, so in Region 1 of the reactor, where $\Delta k_1 > \Delta k_o$, the region of stability is attained later than for the case of uniform Δk_o insertion. However, the flux in the central

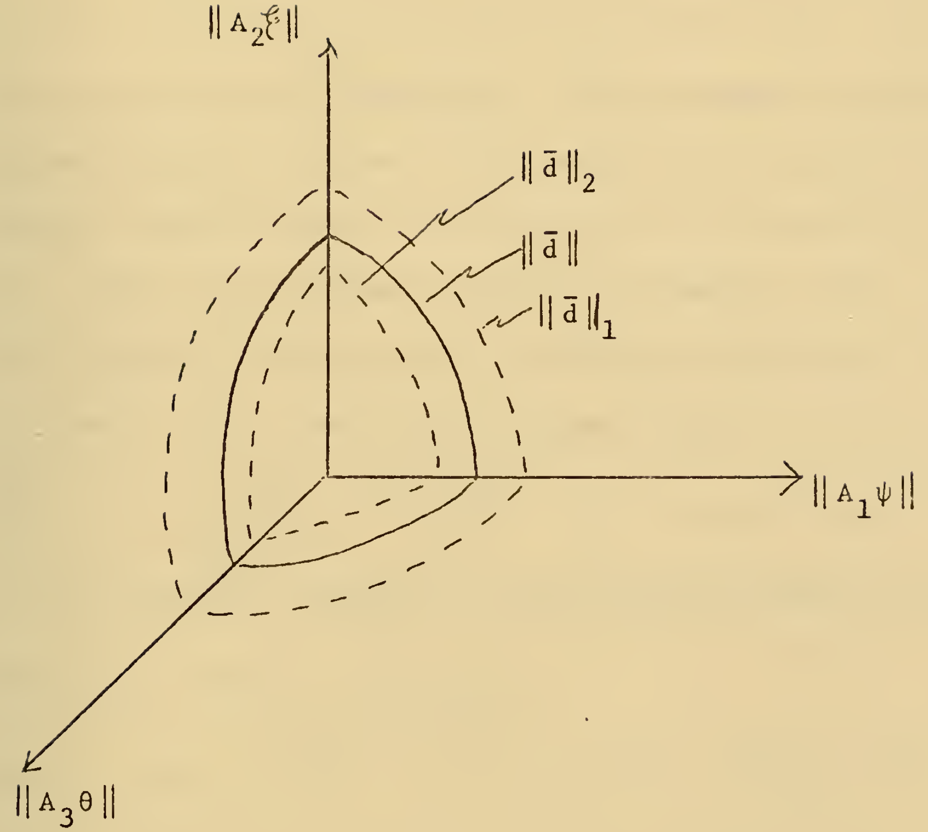

Figure 17. Spherical surfaces for the case of space-dependent Δk_o insertion.

region of the reactor plays a more important role in the transient behavior of the reactor; consequently, the space dependent reactivity insertion, with $\Delta k_2 < \Delta k_o$, is safer than the uniform Δk_o insertion.

3. <u>One-Group Delayed Neutron, General Reactivity Input</u>

For this case, Equations (37), (46), (57) and (66) are the governing equations. The feedback coefficients, γ, γ_D and γ_E, are assumed to be negatives; γ_D and γ_E are the Doppler and Expansion coefficients of reactivity, respectively. The same Liapunov function, Equation (122), is used and a similar development previously discussed leads to the following equation, in analogy to Equation (136):

$$\frac{dV}{d\tau} \leq -\varepsilon^2 \Sigma_f{}^2 \alpha \left\{ \int_{-W}^{W} [a_1 \psi^2 + a_2 \mathcal{C}^2 + a_3 \theta^2 + (1-\beta)\psi^2 F(\theta) + \right.$$

$$\left. + a_4 \quad \psi F(\theta) \; a_5 \theta \psi F(\theta) - (1+\Delta k_o) a_4 \mathcal{C} \psi - (1+\Delta k_o) a_5 \theta \psi] dy \right\} \quad (173)$$

where the feedback term is:

$$F(\theta) = |\gamma|\theta + \frac{|\gamma_D| R^n}{1-n} [T^{1-n} - T_o{}^{1-n}] + \frac{|\gamma_E| P^m}{1+m} [T^{1+m} - T_o{}^{1+m}] \quad (174)$$

here R and P are constants equal to 300°K. $F(\theta)$ is positive for all values of $n > 0$ and $m > 0$. It is noted that the Doppler feedback is the primary feedback mechanism in large, ceramic fueled fast reactors, and expansion is usually the dominant feedback in small metal fueled fast reactors. For $n=1$, Equation (66a) is used. A different approach, more conservative, is used to analyze Equation (173). The expression inside the curly brackets in Equation (173) has to be positive definite in order to ensure asymptotic stability. With

this in mind, a comparison of the orders of magnitude among the terms is carried out.

$$\frac{dV}{d\tau} \leq - \varepsilon^2 \Sigma_f{}^2 \alpha \left\{ \int_{-W}^{W} [a_1 \psi^2 + \mathcal{C}\{a_2 \mathcal{C} - (1+\Delta k_o) a_4 \psi\} + \right.$$

$$\left. + \theta \{a_3 \theta - (1+\Delta k_o) a_5 \psi\} + \psi F(\theta)\{(1-\beta)\psi + a_4 \mathcal{C} + a_5 \theta\}] dy \right\} \quad (175)$$

In order to get a positive definite function inside the curly brackets, the following conditions have to be satisfied:

(a) $a_1 > 0$ which yields $\Delta k_o < \frac{\beta}{1-\beta}$

(b) $a_2 \mathcal{C} - (1+\Delta k_o)\, a_4 \psi > 0$ which yields

$$\frac{\psi}{\mathcal{C}} < \frac{a_2}{(1+\Delta k_o) a_4}$$

and

(c) $a_3 \theta - (1+\Delta k_o) a_5 \psi > 0$ which yields

$$\frac{\psi}{\theta} < \frac{a_3}{(1+\Delta k_o) a_5}$$

Typical values for the feedback coefficients are: $\gamma = 10^{-4}/°K$, $\gamma_D = 7 x 10^{-6}/°K$, $\gamma_E = 9.5 \times 10^{-6}/°K$. Evaluating these conditions for a typical fast reactor, with a Δk_o = 0.156\$, it is obtained that:

(a) $\Delta k_o < 1.00331\$$

(b) $\frac{\psi}{\mathcal{C}} < 3634.0$

(c) $\frac{\psi}{\theta} < 4.203 \times 10^{11}$

These three conditions have to be satisfied simultaneously. The domain of stability is shown in Figures 18a, 18b and 18c. It can be seen that the domain of stability is very large when the coefficients of reactivity are negative.

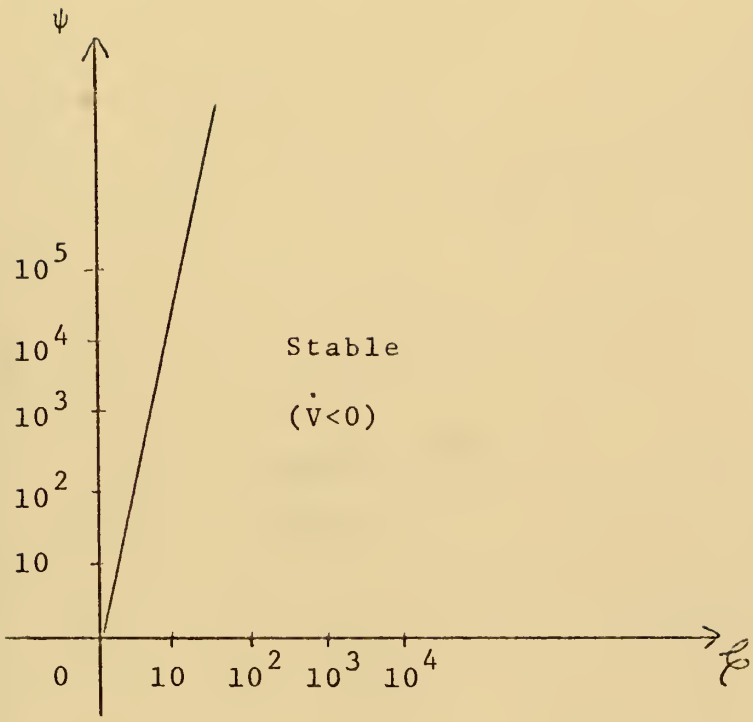

Figure 18a. Stability domains for the distributed parameter reactor system after a general reactivity insertion.

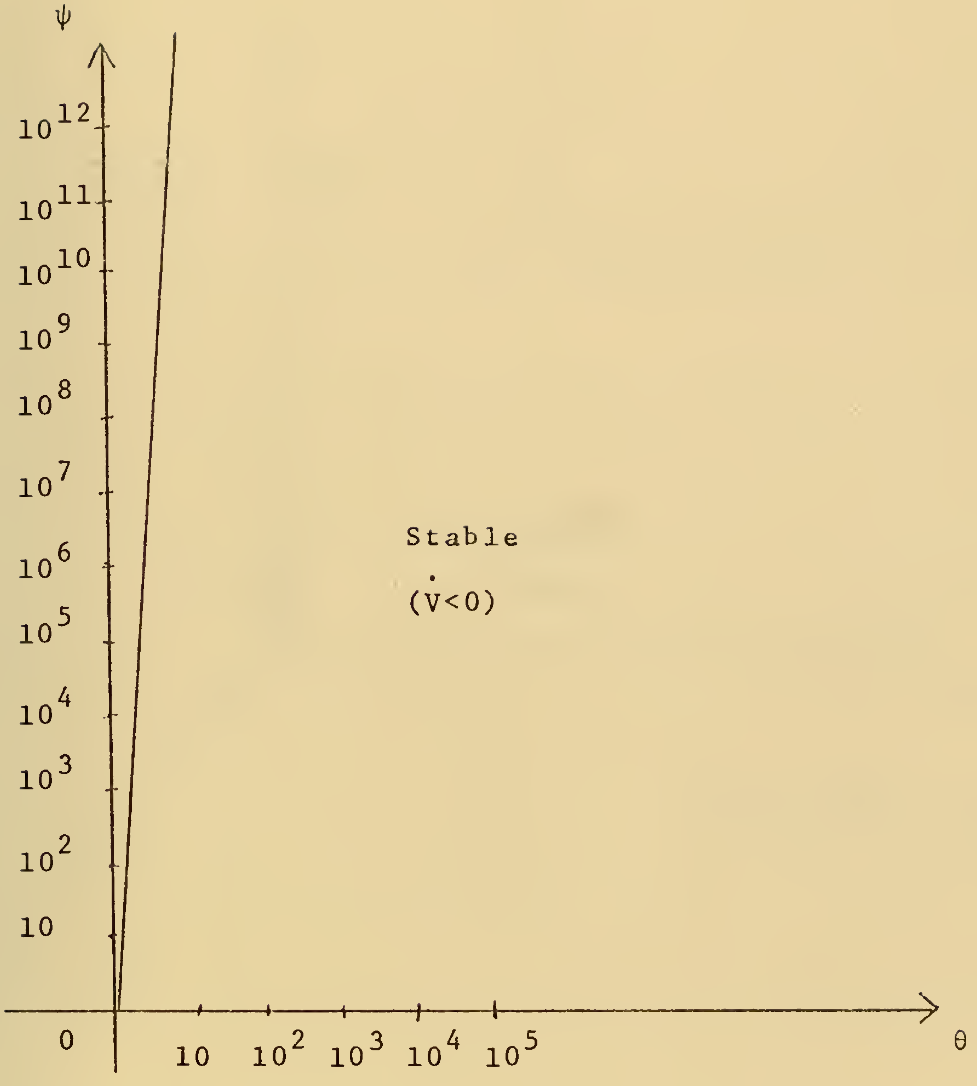

Figure 18b. Stability domains for the distributed parameter reactor system after a general reactivity insertion.

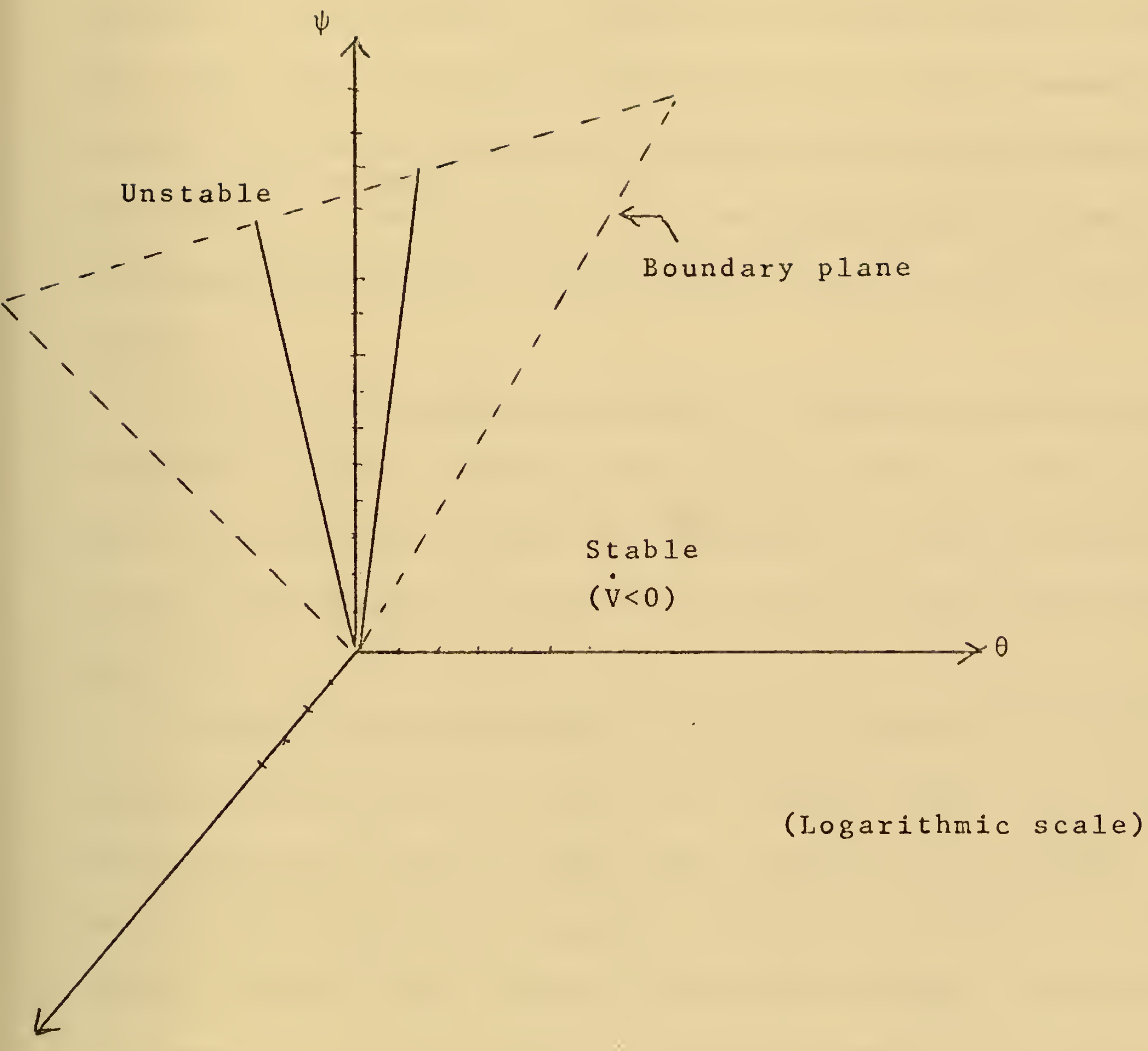

Figure 18c. Stability domains for the distributed parameter reactor system after a general reactivity insertion.

IV. CONCLUSIONS

Liapunov's Second Method has proven to be a useful method to find stability domains in Nuclear Reactor Control. The fact that a Liapunov Function is not unique makes the choice of such a function to depend upon the experience and ingenuity of the analyst. Once an appropriate choice of the Liapunov Function has been made practical results can be found.

For the Point-Kinetics Model, a Liapunov Function was obtained without imposing an a priori bound, as most of the conventional methods do. Domains of stability are presented for different cases using typical data from thermal reactors.

The Distributed-Parameter Reactor System was studied using the concept of a norm, an a priori bound, and the variation with time of this norm, defined in a Hilbert space, was analyzed to obtain a domain of stability. Data from a typical fast reactor was used to define the spherical surface of stability and to estimate a mean value flux, in space and time, when this surface is reached. The case of space dependent Δk_o insertion, with less reactivity in the central region, was found to be safer than the case of uniform Δk_o input.

The linear theory stability condition appeared throughout the analyses giving confidence to the obtained results,

in addition to that condition and additional condition(s) appeared in each case, required by the nonlinear theory.

The inclusion of the multigroup delayed neutrons is recommended for further improvement of this analysis.

APPENDIX A

THE EIGENVALUE INEQUALITY

Let $\lambda > 0$ be the least eigenvalue of the system

$$\frac{d^2\psi}{dy^2} + \lambda\psi = 0 \qquad \text{(A1)}$$

with boundary conditions: (a) $\psi(\pm W) = 0$

the following inequality can be stated:

$$\lambda \int_{-W}^{W} \psi^2 dy \leq \int_{-W}^{W} \left(\frac{d\psi}{dy}\right)^2 dy \qquad \text{(A2)}$$

Proof of this inequality can be done using the calculus of variations.

Let

$$\int_{-W}^{W} (\psi'^2 - \lambda\psi^2) dy \geq 0 \qquad \text{(A3)}$$

Applying Euler's equation: $\frac{d}{dy} F_{\psi'} = F_{\psi}$ (A4)

Equation (A1) is obtained.

Solving the eigenvalue problem λ is obtained to be:

$$\lambda = \frac{\pi^2}{4W^2} = B^2L^2 \qquad \text{(A5)}$$

Then

$$\int_{-W}^{W} \left(\frac{d\psi}{dy}\right)^2 dy \geq B^2L^2 \int_{-W}^{W} \psi^2 dy \qquad \text{(A6)}$$

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ABSTRACT

Any change in neutronic properties in a reactor operating at steady state will result in a change in the equilibrium neutron flux and hence, the power of the reactor. A main cause for a change in neutronic properties is the high temperature attained in a reactor, which produces a feedback in the reactor operation. The response of the reactor to a particular feedback is analyzed by using Liapunov's Second Method to specify stability regimes. Both, the point-kinetics model and the distributed parameters system are analyzed.

Data from a typical thermal and fast reactors is used to specifically determine the stability domains.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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